AN INVERSE APPROACH TO THE CENTER-FOCUS PROBLEM FOR POLYNOMIAL DIFFERENTIAL SYSTEM WITH HOMOGENOUS NONLINEARITIES

JAUME LLIBRE[1], RAFAEL RAMÍREZ[2] AND VALENTÍN RAMÍREZ[3]

Abstract. We consider polynomial vector fields of the form

$$\mathcal{X} = (-y + X_m)\frac{\partial}{\partial x} + (x + Y_m)\frac{\partial}{\partial y},$$

where $X_m = X_m(x,y)$ and $Y_m = Y_m(x,y)$ are homogenous polynomials of degree m. It is well–known that $\mathcal{X}$ has a center at the origin if and only if $\mathcal{X}$ has an analytic first integral of the form

$$H = \frac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j,$$

where $H_j = H_j(x,y)$ is a homogenous polynomial of degree j.

The classical center-focus problem already studied by H. Poincaré consists in distinguishing when the origin of $\mathcal{X}$ is either a center or a focus. In this paper we study the inverse center-focus problem. In particular for a given analytic function H defined in a neighborhood of the origin we want to determine the homogenous polynomials X_m and Y_m in such a way that H is a first integral of $\mathcal{X}$ and consequently the origin of $\mathcal{X}$ will be a center. Moreover, we study the case when

$$H = \frac{1}{2}(x^2 + y^2)\left(1 + \sum_{j=1}^{\infty} \Upsilon_j\right),$$

where Υ_j is a convenient homogenous polynomial of degree j for $j \geq 1$.

The solution of the inverse center problem for polynomial differential systems with homogenous nonlinearities, provides a new mechanism to study the center problem, which is equivalent to Liapunov's Theorem and Reeb's criterion.

1. Introduction

Let

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial},$$

be the real planar polynomial vector field associated to the real planar polynomial differential system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1}$$

2010 *Mathematics Subject Classification.* 34C07.

Key words and phrases. center-foci problem, analytic planar differential system, Liapunov's constants, isochronous center, uniform isochronous center, holomorphic isochronous center, Darboux's first integral, weak condition for a center.