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An inverse approach to the center problem

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Abstract We consider analytic or polynomial vector fields of the form $\mathcal{X} = (-y + X)\dfrac{\partial}{\partial x} + (x + Y)\dfrac{\partial}{\partial y}$, where $X = X(x, y))$ and $Y = Y(x, y))$ start at least with terms of second order. It is well-known that $\mathcal{X}$ has a center at the origin if and only if $\mathcal{X}$ has a Liapunov–Poincaré local analytic first integral of the form $H = \dfrac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j$, where $H_j = H_j(x, y)$ is a homogenous polynomial of degree j. The classical center-focus problem already studied by Poincaré consists in distinguishing when the origin of $\mathcal{X}$ is either a center or a focus. In this paper we study the inverse center problem, i.e. for a given analytic function H of the previous form defined in a neighborhood of the origin, we determine the analytic or polynomial vector field $\mathcal{X}$ for which H is a first integral. Moreover, given an analytic function $V = 1+\sum_{j=1}^{\infty} V_j$ in a neighborhood of the origin, where V_j is a homogenous polynomial of degree j, we determine the analytic or polynomial vector field $\mathcal{X}$ for which V is a Reeb inverse integrating factor. We study the particular case of centers which have a local analytic first integral of the form $H = \dfrac{1}{2}(x^2 + y^2)\left(1 + \sum_{j=1}^{\infty} \Upsilon_j\right)$, in a neighborhood of the origin, where Υ_j is a homogenous polynomial of degree j for $j \geq 1$. These centers are called weak centers, they contain the uniform isochronous centers and the isochronous holomorphic centers, but they do not coincide with the class of isochronous centers. We have characterized the expression of

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