THE 16TH HILBERT PROBLEM RESTRICTED TO CIRCULAR ALGEBRAIC LIMIT CYCLES

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Abstract. We prove the following two results.

First every planar polynomial vector field of degree S with S invariant circles is Darboux integrable without limit cycles.

Second a planar polynomial vector field of degree S, admits at most $S-1$ invariant circles which are algebraic limit cycles.

In particular we solve the 16th Hilbert problem restricted to algebraic limit cycles given by circles, because a planar polynomial vector field of degree S has at most $S-1$ algebraic limit cycles given by circles, and this number is reached.

1. Introduction and statement of the main results

Let $\mathbb{R}[x,y]$ be the ring of all real polynomials in the variables x and y. Assume that P, $Q \in \mathbb{R}[x,y]$ such that P and Q are coprime in $\mathbb{R}[x,y]$. Consider the set Σ of all planar real polynomial vector fields

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial x},$$

associated to the differential polynomial systems

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y).$$

of degree $m = \max\{\text{degP}, \text{degQ}\}$, here the dot denotes derivative respect to the time t.

Let U be an open and dense set in $\mathbb{R}^2$. We say that a non-constant C^1 function $H : U \to \mathbb{R}$ is a *first integral* of the polynomial vector field $\mathcal{X}$ on U, if $H(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ of $\mathcal{X}$ is defined on U. Clearly H is a first integral of $\mathcal{X}$ on U if and only if $\mathcal{X}H = 0$ on U.

Let $g = g(x,y) \in \mathbb{R}[x,y]$. Then Let $g = 0$ is an *invariant algebraic curve* of $\mathcal{X}$ if

$$\mathcal{X}g = P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} = Kg,$$

where $K = K(x,y)$ is a polynomial of degree at most $m-1$, which is called the *cofactor* of $g = 0$. If the polynomial g is irreducible in $\mathbb{R}[x,y]$, then we say that the invariant algebraic curve $g = 0$ is *irreducible* and that its *degree* is the degree of the polynomial g.

We recall that a *limit cycle* of a polynomial vector field $\mathcal{X}$ is an isolated periodic orbit in the set of all periodic orbits of $\mathcal{X}$. An *algebraic limit cycle of degree* n of

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$\mathcal{X}$ is an oval of an irreducible invariant algebraic curve $g = 0$ of degree n, which is a limit cycle of $\mathcal{X}$.

Hilbert in [3] asked: *Is there an upper bound for the maximum number of limit cycles of any polynomial vector field with a given degree*? This is a version of the second half part of the *Hilbert's 16–th problem*. This problem remains open, see for more information [4, 5].

A simpler version of the second part of the 16–th Hilbert's problem restricted to algebraic limit cycles can be stated as follows: Consider the set Σ'_m of all real polynomial vector fields $\mathcal{X}$ of degree m having real invariant algebraic curves. *Is there an upper bound on the maximum number of algebraic limit cycles of any polynomial vector field of* Σ'_m? (see [6, 7]).

There is the following conjecture (see [7]) about the maximum number of algebraic limit cycles of polynomial vector fields with a given degree.

Conjecture 1. *The maximum number of algebraic limit cycles that a polynomial vector field of degree* $m \geq 2$ *can have is* $1 + (m-1)(m-2)/2$.

Conjecture 1 has been proved when the invariant algebraic curves of the polynomial vector fields satisfy some generic properties see [7], see also [6, 8, 12].

Let $f_i, g_j, h_j \in \mathbb{R}[x, y]$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Then the (multi–valued) function

$$|f_1|^{\lambda_1} \cdots |f_p|^{\lambda_p} e^{\mu_1 g_1/h_1} \cdots e^{\mu_q g_q/h_q}$$

with $\lambda_i, \mu_j \in \mathbb{C}$ is called a *(generalized) Darboux function.*

A *configuration of circles* is a finite collection of disjoint circles. We say that a configuration of circles is *realizable as algebraic limit cycles* if there exists a polynomial vector field such that all its circles of the given configuration are algebraic limit cycles of the vector field.

A *nest* of r circles is formed by a finite numbers $C_1, \ldots, C_r$ of circles such that its configuration is homomorphic to the configuration $x^2 + y^2 - j^2 = 0$, for $j = 1, \ldots, r$.

Let g_1 and g_2 be functions defined in an open subset $U \subseteq \mathbb{R}^2$. We define the Jacobian matrix of g_1 and g_2 as

$$J = \begin{pmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} \end{pmatrix}.$$

The Jacobian of J, i.e. the determinant of J is denoted here by

$$|J| := \{g_1, g_2\}.$$

Our main results are the following.

Theorem 2. *We consider the polynomial differential system*

$$\begin{aligned} \dot{x} = & \quad -\lambda_{S+1} \prod_{m=1}^{S} g_m + \sum_{j=1}^{S} \lambda_j \left(\prod_{\substack{m=1 \\ m \neq j}}^{S} g_m \right) \{g_j,\, x\} = P(x, y), \\ \dot{y} = & \quad \lambda_{S+2} \prod_{m=1}^{S} g_m + \sum_{j=1}^{S} \lambda_j \left(\prod_{\substack{m=1 \\ m \neq j}}^{S} g_m \right) \{g_j,\, y\} = Q(x, y). \end{aligned} \tag{1}$$

where λ_j for $j = 1, \ldots, S+2$ are arbitrary polynomials. This system has $g_j = 0$ as invariant algebraic curves for $j = 1, \ldots, S$. We assume that $\{g_1, g_2\} \not\equiv 0$. Then the generalized Darboux function $F = e^{\tau} \prod_{j=1}^{S} |g_j|^{\lambda_j}$ is a first integral of the polynomial differential system (1) if and only if $\lambda_{S+1} = \dfrac{\partial \tau}{\partial y}$, $\lambda_{S+2} = \dfrac{\partial \tau}{\partial x}$, where $\tau = \tau(x, y)$ is an arbitrary polynomial and $\lambda_1, \ldots, \lambda_S$ are constants.

Theorem 2 is proved in section 3.

Theorem 3. *A polynomial vector field $\mathcal{X}$ of degree S with S invariant circles is Darboux integrable, and the invariant circles are not limit cycles.*

Theorem 3 is proved in section 4

Theorem 4. *For a polynomial vector field $\mathcal{X}$ of degree S the maximum number of algebraic limit cycles given by circles is at most $S - 1$. Moreover this upper bound is reached.*

Theorems 4 is proved in section 5. We note that Theorem 4 shows that polynomial vector fields of degree S can have at most $S - 1$ circles as limit cycles.

Theorem 3 and 4 provide the solution of the 16th Hilbert problem restricted to algebraic limit cycles given by circles.

Some basic results that we shall need for proving Theorems 2, 3 and 4 are stated in section 2.

2. Preliminary results

The next result is the Proposition 2.1 of [10], see also the Corollary 1.3.4 of [9]. Here we provide a proof of it because some arguments of the proof will be used later on.

Theorem 5. *Let $g_j = g_j(x, y)$ for $j = 1, 2, \ldots, S$ with $S \geq 2$ polynomials such that at least two of them (that without loss of generality we can assume that they are g_1 and g_2) satisfy $\{g_1, g_2\} \not\equiv 0$. Then a polynomial differential system having the curves $g_j = 0$ as invariant algebraic curves with cofactors $K_j = \{g_1, g_2\}\mu_j$ for $j = 1, \ldots, S$ respectively, and satisfying*

$$\mu_j g_j \{g_1, g_2\} = \mu_1 g_1 \{g_j, g_2\} + \mu_2 g_2 \{g_1, g_j\}, \tag{2}$$

for $j = 3, \ldots, S$, can be written as

$$\begin{aligned} \dot{x} &= \mu_1 g_1 \{x, g_2\} + \mu_2 g_2 \{g_1, x\} = P(x, y), \\ \dot{y} &= \mu_1 g_1 \{y, g_2\} + \mu_2 g_2 \{g_1, y\} = Q(x, y). \end{aligned} \tag{3}$$

where μ_j for $j = 1, \ldots, S$ are arbitrary rational functions such that P and Q are polynomials.

Proof. If the polynomial planar vector field associated to system (3) admits the curves $g_j = 0$ for $j = 1, \ldots, S$ as invariant algebraic curves then

$$\begin{aligned}
\frac{\partial g_1}{\partial x} P + \frac{\partial g_1}{\partial y} Q = &\quad g_1 K_1, \\
\frac{\partial g_2}{\partial x} P + \frac{\partial g_2}{\partial y} Q = &\quad g_2 K_2, \\
\frac{\partial g_j}{\partial x} P + \frac{\partial g_j}{\partial y} Q = &\quad g_j K_j \Longleftrightarrow \{g_j, y\}P + \{x, g_j\}Q = g_j K_j,
\end{aligned}$$

for $j = 3, \ldots, S$. By solving the two first equations with respect to P and Q and by considering that the matrix coefficient is the matrix J with determinant $\{g_1, g_2\}$, we obtain

$$\begin{aligned}
P = &\quad \frac{K_1 g_1}{\{g_1, g_2\}}\{x, g_2\} + \frac{K_2 g_2}{\{g_1, g_2\}}\{g_1, x\}, \\
Q = &\quad \frac{K_1 g_1}{\{g_1, g_2\}}\{y, g_2\} + \frac{K_2 g_2}{\{g_1, g_2\}}\{g_1, y\}.
\end{aligned} \tag{4}$$

By inserting P and Q in the last $S - 2$ equations we deduce

$$\begin{aligned}
&K_1 g_1 \left(\{x, g_2\}\{g_j, y\} + \{y, g_2\}\{x, g_j\}\right) + K_2 g_2 \left(\{g_1, x\}\{g_j, y\} + \{g_1, y\}\{x, g_j\}\right) \\
&= \{g_1, g_2\} g_j K_j.
\end{aligned}$$

In view of the identity

$$\{f, p\}\{g, q\} + \{f, q\}\{p, g\} = \{f, g\}\{p, q\}, \tag{5}$$

for arbitrary C^1 functions f, g, q, p. we finally obtain

$$K_1 g_1 \{g_j, g_2\} + K_2 g_2 \{g_1, g_j\} = \{g_1, g_2\} g_j K_j.$$

After the change $K_j = \{g_1, g_2\}\mu_j$ we finally obtain the proof of the theorem. □

Proposition 6. *Let $g_j = g_j(x, y)$ for $j = 1, 2, \ldots, S$ with $S \geq 2$ polynomials such that at least two of them (that without loss of generality we can assume that they are g_1 and g_2) satisfy $\{g_1, g_2\} \not\equiv 0$. Then the differential system* (3) *satisfying* (2) *and the differential system* (1) *are equivalent.*

Proof. For $n = 1, \ldots, S$ we define $K_n = \{g_1, g_2\}\mu_n$ as follows

$$K_n g_n = \sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m \neq j}}^{S} g_m \right) \{g_j,\, g_n\}, \tag{6}$$

where $g_{S+1} = y$ and $g_{S+2} = x$ and $\lambda_1, \ldots, \lambda_{S+2}$ are arbitrary rational functions such that K_n be a polynomial.

Substituting $K_j g_j$, $K_1 g_1$ and $K_2 g_2$ from (6) into (2) we obtain that equalities (2) become

$$\sum_{l=1}^{S+2} \lambda_l \left(\prod_{\substack{m=1 \\ m \neq l}}^{S} g_m \right) \left(\{g_l,\, g_j\}\{g_1, g_2\} + \{g_1,\, g_l\}\{g_j, g_2\} + \{g_2,\, g_l\}\{g_1, g_j\}\right) = 0.$$

This equality holds in view of identity (5).

Assume that we have system (4) satisfying (2) with $K_n = \{g_1, g_2\}\mu_n$ given in (6). Then, substituting the K_n in (4) which is equivalent to (3) we have

$$\begin{aligned} P = & \frac{\sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_1,\, g_j\}}{\{g_1, g_2\}} \{x, g_2\} + \frac{\sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_2,\, g_j\}}{\{g_1, g_2\}} \{g_1, x\} \\ = & \frac{\sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right)}{\{g_1, g_2\}} \left(\{g_1,\, g_j\}\{x, g_2\} + \{g_2,\, g_j\}\{g_1, x\} \right), \\ Q = & \frac{\sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_1,\, g_j\}}{\{g_1, g_2\}} \{y, g_2\} + \frac{\sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_2,\, g_j\}}{\{g_1, g_2\}} \{g_1, y\} \\ = & \frac{\sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right)}{\{g_1, g_2\}} \left(\{g_1,\, g_j\}\{y, g_2\} + \{g_2,\, g_j\}\{g_1, y\} \right). \end{aligned}$$

and using the relation (5) we have

$$\begin{aligned} P = & \frac{\sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right)}{\{g_1, g_2\}} \{g_1, g_2\}\{g_j, x\} = \sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_j, x\} \\ = & -\lambda_{S+1} \prod_{n=1}^{S} g_n + \sum_{j=1}^{S} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_j, x\}, \\ Q = & \frac{\sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right)}{\{g_1, g_2\}} \{g_1, g_2\}\{g_j, y\} = \sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_j, y\} \\ = & \lambda_{S+2} \prod_{n=1}^{S} g_n + \sum_{j=1}^{S} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_j, y\}. \end{aligned}$$

Thus by considering that $\{x, y\} = 1$, $\{x, x\} = 0$ and $\{y, y\} = 0$, then the differential system (3) coincide with system (1).

Now we consider system (1) written in the form

$$\dot{x} = \sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_j, x\}, \quad \dot{y} = \sum_{j=1}^{S+2} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_j, y\},$$

where $g_{S+1} = y$ and $g_{s+2} = x$. Thus we easily obtain the relations (6).

We consider the equalities

$$-\lambda_{S+1} \prod_{m=1}^{S} g_m + \sum_{j=1}^{S} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_j,\, x\} = \frac{K_1 g_1}{\{g_1, g_2\}} \{x, g_2\} + \frac{K_2 g_2}{\{g_1, g_2\}} \{g_1, x\},$$

$$\lambda_{S+2} \prod_{m=1}^{S} g_m + \sum_{j=1}^{S} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right) \{g_j,\, y\} = \frac{K_1 g_1}{\{g_1, g_2\}} \{y, g_2\} + \frac{K_2 g_2}{\{g_1, g_2\}} \{g_1, y\},$$

or equivalently

$$\lambda_1 \{\log|g_1|, x\} + \lambda_2 \{\log|g_2|, x\} = \lambda_{S+1} - \sum_{j=3}^{S} \lambda_j \{\log|g_j|,\, x\}$$
$$+ \frac{1}{g} \left(\frac{K_1 g_1}{\{g_1, g_2\}} \{x, g_2\} + \frac{K_2 g_2}{\{g_1, g_2\}} \{g_1, x\} \right),$$
$$\lambda_1 \{\log|g_1|, y\} + \lambda_2 \{\log|g_2|, y\} = -\lambda_{S+2} - \sum_{j=3}^{S} \lambda_j \{\log|g_j|,\, y\}$$
$$+ \frac{1}{g} \left(\frac{K_1 g_1}{\{g_1, g_2\}} \{x, g_2\} + \frac{K_2 g_2}{\{g_1, g_2\}} \{g_1, y\} \right),$$

where $g = \prod_{m=1}^{S} g_m$. Thinking this last system as a linear system in the variables λ_1 and λ_2 it can be solved because its determinant is $\{g_1, g_2\}/g_1 g_2$. So the differential system (3) can be obtained from (1). Moreover, since the cofactor of the curve $g_n = 0$ for $n = 1, \ldots, S$ in (1) is the K_n defined in (6), the condition (2) holds. Hence the theorem is proved. □

We observe that (1) is a particular case of the equations given in Theorem 1.6.1 of [6].

We remark that if the statement of Proposition 6 we have $\{g_1,\, g_2\} \equiv 0$, then the differential systems(1) and (3) are not equivalent. Indeed if the given circles are concentric, i.e.

$$g_j = x^2 + y^2 - r_j^2 = 0, \quad j = 1, \ldots, S,$$

with $0 < r_1 < r_2 < \ldots < r_S$, then $\{g_1,\, g_2\} \equiv 0$, and system (1) takes the form

$$\dot{x} = -\lambda_{S+1} g - y\nu, \quad \dot{y} = \lambda_{S+2} g + x\nu,$$

where $g = \prod_{m=1}^{S} g_m$ and $\nu = \prod_{m=1}^{S} g_m \sum_{j=1}^{S} \lambda_j \left(\prod_{\substack{m=1 \\ m\neq j}}^{S} g_m \right)$, and system (3) cannot be defined.

3. Proof of Theorem 2

Proof of Theorem 2. Assume that the λ_j's for $j = 1, \ldots, S$ of the differential system (1) are constant and $\lambda_{S+1} = \dfrac{\partial \tau}{\partial y} = \tau_y$ and $\lambda_{S+2} = \dfrac{\partial \tau}{\partial x} = \tau_x$, being $\tau = \tau(x, y)$ a polynomial. Then system (1) admits the representation

$$\begin{aligned}
\dot{x} = & \quad g\left(-\frac{\partial \tau}{\partial y} + \sum_{j=1}^{S} \lambda_j \{\log|g_j|,\, x\}\right) = \frac{g}{F}\left(-\frac{\partial \tau}{\partial y} F + F \sum_{j=1}^{S} \lambda_j \{\log|g_j|,\, x\}\right) \\
= & \quad \frac{g}{F}\left(-\frac{\partial \tau}{\partial y} F + \sum_{j=1}^{S} \lambda_j \{F,\, x\}\right) = g\{\log\, F,\, x\}, \\
\dot{y} = & \quad g\left(\frac{\partial \tau}{\partial x} + \sum_{j=1}^{S} \lambda_j \{\log|g_j|,\, y\}\right) = \frac{g}{F}\left(\frac{\partial \tau}{\partial x} F + F \sum_{j=1}^{S} \lambda_j \{\log|g_j|,\, y\}\right) \\
= & \quad \frac{g}{F}\left(\frac{\partial \tau}{\partial x} F + \sum_{j=1}^{S} \lambda_j \{F,\, y\}\right) = g\{\log\, F,\, y\},
\end{aligned}$$

where $F = e^{\tau} \prod_{m=1}^{S} |g_m|^{\lambda_m}$, $g = \prod_{m=1}^{S} g_m$. and we have used

$$\begin{aligned}
F_x = & \quad \tau_x F + e^{\tau}\left(\sum_{j=1}^{S} \lambda_j g_j^{\lambda_j - 1} \{g_j, y\} \prod_{\substack{m=1 \\ m\neq j}}^{S} |g_m|^{\lambda_m}\right) \\
= & \quad F\left(\tau_x + \sum_{j=1}^{S} \lambda_j \{\log|g_j|, y\}\right), \\
F_y = & \quad \tau_y F - e^{\tau}\left(\sum_{j=1}^{S} \lambda_j g_j^{\lambda_j - 1} \{g_j, x\} \prod_{\substack{m=1 \\ m\neq j}}^{S} |g_m|^{\nu_m}\right) \\
= & \quad -F\left(-\tau_y + \sum_{j=1}^{S} \lambda_j \{\log|g_j|, x\}\right),
\end{aligned} \tag{7}$$

and

$$
\begin{aligned}
P(x,y) = &\quad -\lambda_{S+1}\prod_{m=1}^{S} g_m + \sum_{j=1}^{S}\lambda_j\left(\prod_{\substack{m=1\\ m\neq j}}^{S} g_m\right)\{g_j,\, x\}\\
= &\quad g\left(-\lambda_{S+1} + \sum_{j=1}^{S}\lambda_j\{\log|g_j|,\, x\}\right),\\
Q(x,y) = &\quad \lambda_{S+2}\prod_{m=1}^{S} g_m + \sum_{j=1}^{S}\lambda_j\left(\prod_{\substack{m=1\\ m\neq j}}^{S} g_m\right)\{g_j,\, y\}\\
= &\quad g\left(\lambda_{S+2} + \sum_{j=1}^{S}\lambda_j\{\log|g_j|,\, y\}\right).
\end{aligned}
\tag{8}
$$

Consequently the function F is a first integral.

Let $F = e^{\tau}\prod_{m=1}^{S}|g_m|^{\nu_m}$ where ν_m for $m = 1,\dots,S$ are constants. Then F is a first integral of the vector field (1). By applying (7) and (8) we have that

$$
\begin{aligned}
\dot{F} = &\quad F_xP + F_yQ\\
= &\quad F\left(\tau_x + \sum_{j=1}^{S}\nu_j\{\log|g|_j, y\}\right)\\
&\quad g\left(-\lambda_{S+1} + \sum_{j=1}^{S}\lambda_j\{\log|g|_j, y\}\right) +\\
&\quad F\left(\tau_y - \sum_{j=1}^{S}\nu_j\{\log|g|_j, x\}\right)\\
&\quad g\left(\lambda_{S+2} + \sum_{j=1}^{S}\lambda_j\{\log|g|_j, x\}\right) +\\
\equiv &\quad 0.
\end{aligned}
$$

This relation holds if $\lambda_j = \nu_j$ and $\lambda_{S+1} = \tau_y$ and $\lambda_{S+2} = \tau_x$. This completes the proof of the theorem. □

Corollary 7. *The vector field*

$$
\dot{x} = \sum_{j=1}^{S}\lambda_j\left(\prod_{\substack{m=1\\ m\neq j}}^{S} g_m\right)\{g_j, x\}, \quad \dot{y} = \sum_{j=1}^{S}\lambda_j\left(\prod_{\substack{m=1\\ m\neq j}}^{S} g_m\right)\{g_j, y\},
\tag{9}
$$

is Darboux integrable if and only if $\lambda_1,\dots,\lambda_S$ are constants.

Proof. It follows easily from Theorem 2. □

4. Proof of Theorem 3

The proof of Theorem 3 follows from the following results.

By Proposition 6 and Theorem 5 we know that any polynomial differential system having the circles

$$g_j(x,y) \equiv (x-a_j)^2 + (y-b_j)^2 - r_j^2 = 0, \quad j = 1, 2, \ldots, S \tag{10}$$

as invariant algebraic curves can be written as system (1), i.e.

$$\begin{aligned} \dot{x} = & \quad -\lambda_{S+1} \prod_{m=1}^{S} g_m - 2 \sum_{j=1}^{S} \lambda_j (y - b_j) \left(\prod_{\substack{m=1 \\ m \neq j}}^{S} g_m \right) = P(x,y), \\ \dot{y} = & \quad \lambda_{S+2} \prod_{m=1}^{S} g_m + 2 \sum_{j=1}^{S} \lambda_j (x - a_j) \left(\prod_{\substack{m=1 \\ m \neq j}}^{S} g_m \right) = Q(x,y), \end{aligned} \tag{11}$$

where $\lambda_1, \ldots, \lambda_{S+2}$ are arbitrary polynomials. We assume that system (37) has degree S, because this is an assumption of Theorem 3.

From now on we shall assume without loss of generality that $a_1 = b_1 = b_2 = 0$.

Proposition 8. *The cofactors of the circles* (10) *for the system* (37) *are*

$$K_j = 4a_2 y \left(\kappa_0^{(j)} + \kappa_1^{(j)} + \ldots + \kappa_{S-3}^{(j)} + \kappa_{S-2} \right), \tag{12}$$

where $\kappa_n^{(j)}$ *are homogenous polynomial of degree* n*, and* κ_{S-2} *is an homogenous polynomial of degree* $S-2$*. Note that the homogenous polynomial of degree* $S-2$ *are the same for all* $j = 1, \ldots, S$*.*

Proof. From the proof of Theorem 5 and from the equivalence between the systems (3) and (1) given in Proposition 6, we have that the cofactor K_j of the invariant circle $g_j = 0$ for $j = 1, \ldots, S$ are of the form $K_j = \{g_1, g_2\}\mu_j$ being μ_j a rational function . Since $\{g_1, g_2\} = 4a_2 y$ and K_j must be a polynomial of degree at most $S-1$, we have that μ_j is a polynomial of degree $S-2$. Then we shall determine the cofactors K_j for $j = 1, \ldots, S$ as follows

$$K_j = 4a_2 y \sum_{n=0}^{S-2} \kappa_n^{(j)} = y \sum_{n+l=0}^{S-2} \kappa_{n,l}^{(j)} x^n y^l, \tag{13}$$

where $\kappa_n^{(j)} = \kappa_n^{(j)}(x,y)$ are homogenous polynomial of degree n. From the conditions (2) it follows that

$$\begin{aligned} & y \left(K_j g_j a_2 + K_1 g_1 (a_j - a_2) - K_2 g_2 a_j \right) \\ & + b_j x (K_2 g_2 - K_1 g_1) + b_j K_1 g_1 a_2 = 0, \end{aligned} \tag{14}$$

for $j = 3, \ldots, S$. Inserting (13) in these equations we obtain that

$$(x^2 + y^2) \left(y \left(a_2 \kappa_{S-2}^{(j)} + (a_j - a_2) \kappa_{S-2}^{(1)} - a_j \kappa_{S-2}^{(2)} \right) + b_j x (\kappa_{S-2}^{(1)} - \kappa_{S-2}^{(2)}) \right) + \ldots = 0,$$

On the other hand, from (2) it follows that

$$
\begin{aligned}
\dot{x} = & \frac{1}{4a_2}(K_1 g_1 - K_2 g_2) \\
= & \; y\left((x^2+y^2)(\kappa^{(1)}_{S-2} - \kappa^{(2)}_{S-2}) + \ldots\right) = P, \\
\dot{y} = & -\frac{1}{4a_2 y}\left(x(K_1 g_1 - K_2 g_2) + a_2 K_1 g_1\right) \\
= & -x\left((x^2+y^2)(\kappa^{(1)}_{S-2} - \kappa^{(2)}_{S-2}) + \ldots\right) \\
& + a_2\left((x^2+y^2)\kappa^{(1)}_{S-2} + \ldots\right) = Q.
\end{aligned} \tag{15}
$$

Hence by considering that $m = \max(\deg P, \deg Q) = S$, we obtain that $\kappa^{(1)}_{S-2} = \kappa^{(2)}_{S-2}$. From (25) we finally deduce that

$$\kappa^{(1)}_{S-2} = \kappa^{(2)}_{S-2} = \ldots = \kappa^{(S)}_{S-2} = \kappa_{S-2}.$$

Consequently (12) is proved. □

Proposition 9. *If the S concentric circles*

$$g_j = x^2 + y^2 - r_j^2 = 0 \quad \textit{for} \quad j = 1, \ldots, S,$$

with $r_1 < r_2 < \ldots < r_S$ are invariant circles of the polynomial system of degree S then this system admits the first integral $F = x^2 + y^2$. Consequently this system has no limit cycles.

Proof. Indeed the polynomial planar vector field with S invariant concentric circles has the form (see formula (1))

$$
\begin{aligned}
\dot{x} = & \; -\lambda_{S+1}\prod_{m=1}^{S} g_m - 2y\sum_{j=1}^{S}\lambda_j\left(\prod_{\substack{m=1\\ m\neq j}}^{S} g_m\right) = -y\nu - \lambda_{S+1}\prod_{m=1}^{S} g_m, \\
\dot{y} = & \; \lambda_{S+2}\prod_{m=1}^{S} g_m + 2x\sum_{j=1}^{S}\lambda_j\left(\prod_{\substack{m=1\\ m\neq j}}^{S} g_m\right) = x\nu + \lambda_{S+2}\prod_{m=1}^{S} g_m.
\end{aligned}
$$

Clearly if this polynomial system has degree S then $\lambda_{S+1} = \lambda_{S+2} = 0$ and λ_j for $j = 1, \ldots, S$ are such that the polynomial ν has degree at most $S-1$. Consequently the most general polynomial planar vector field of degree S with S invariant concentric circles takes the form

$$\dot{x} = -y\nu, \quad \dot{y} = x\nu.$$

This system admits the first integral $x^2 + y^2$. □

Proposition 10. *The quadratic vector fields with two invariant circles are rational integrable.*

Proof. For the case when $m = 2$ we always can consider that the given invariant circles are

$$g_1 = x^2 + y^2 - r_1^2 = 0, \quad g_2 = (x - a_2)^2 + y^2 - r_2^2 = 0, \quad a_2 \neq 0.$$

In view of Proposition 8 we obtain that the cofactors of the given circles are $K_1 = K_2 = 4a_2qy$. Then from (3) the quadratic vector field is

$$\begin{aligned}\dot{x} = &\quad -qy\left(2a_2x + r_2^2 - r_1^2 - a_2^2\right),\\ \dot{y} = &\quad 2q\left(a_2(y^2 - x^2) + (a_2^2 + r_1^2 - r_2^2)x - a_2r_1^2\right),\end{aligned}$$

Consequently this quadratic system has the rational first integral $F = g_1/g_2$. □

Proposition 11. *The cubic vector fields with the three invariant circles*

$$\begin{aligned} g_1 = &\quad (x-a_1)^2 + y^2 - r_1^2 = 0,\\ g_2 = &\quad (x-a_2)^2 + y^2 - r_2^2 = 0,\\ g_3 = &\quad (x-a_3)^2 + (y-b_3)^2 - r_3^2 = 0,\\ \{g_1, g_2\} = &\quad 4(a_2-a_1)y, \quad a_1 \neq a_2,\end{aligned} \tag{16}$$

is the zero vector field if $b_3 \neq 0$, and Darboux integrable if $b_3 = 0$ with the first integral $F = \prod_{j=1}^{3} |g_j|^{\lambda_j}$, where

$$\lambda_1 = \lambda_0(a_2 - a_3), \quad \lambda_2 = \lambda_0(a_3 - a_1), \quad \lambda_3 = \lambda_0(a_1 - a_2), \tag{17}$$

and λ_0 is a constant.

Proof. Indeed the cofactor of the circles are polynomials of degree 2 which we determine as follows

$$K_j = 4a_2y\left(C_j + Ax + By\right),$$

for $j = 1, 2, 3$. Here we use Proposition 8. Thus the equation

$$K_1g_1\{g_2, g_3\} + K_2g_2\{g_3, g_1\} + K_3g_3\{g_1, g_2\} = 0,$$

is a polynomial of degree 5. This polynomial is a zero polynomial if and only if the constants C_1, C_2, C_3, A and B are such that

$$\begin{aligned} b_3\left(A_1(a_2^2 - a_3^2 + r_1^2) + a_2(C_1 - 2C_2)\right) &= 0,\\ b_3r_1^2a_2C_1 &= 0,\\ b_3\left(C_2(r_2^2 - a_2^2) - C_1r_1^2 + Ar_1^2a_2\right) &= 0,\\ b_3\left(C_1 - C_2 + a_2A\right) &= 0,\\ a_2C_3 + a_3C_1 - a_3C_2 - a_2b_3B &= 0\\ B\left(a_2b_3^2 - a_3r_1^2 - a_2r_3^2 + a_2r_1^2 + a_2a_3^2 + a_3r_2^2 - a_2^2a_3\right)&\\ +b_3\left(a_2C_1 - a_2C_3\right) &= 0,\\ A\left(a_2r_1^2 + a_2b_3^2 - a_2r_3^2 + a_3r_2^2 - a_3r_1^2 - a_2^2a_3 + a_2a_3^2\right)&\\ +a_2a_3\left(C_3 - C_2\right) &= 0,\\ b_3B\left(a_2^2 + r_1^2 - r_2^2\right) &= 0,\\ r_1^2C_1\left(a_2 - a_3\right) + a_3C_2\left(r_2^2 - a_2^2\right) - a_2b_3Br_1^2&\\ +C_3a_2\left(b_3^2 - r_3^2 + a_3^2\right) &= 0,\end{aligned} \tag{18}$$

Thus we have a system of 9 equations linear with respect to the 5 variables C_1, C_2, C_3, A and B. After some computations we obtain:

(i) If $b_3 \neq 0$ then the unique solutions are

$$A = B = C_1 = C_2 = C_3 = 0,$$

thus the cofactors are all zero and consequently the vector field is a zero vector field.

(ii) If $b_3 = 0$ and $(a_3 - a_2)a_3 \neq 0$, then the linear system is formed by 4 equations admits a non trivial solutions

$$\begin{aligned} C_1 = &\quad \lambda_0 a_2 a_3 \left(r_3^2 - r_2^2 + a_2^2 - a_3^2\right), \\ C_2 = &\quad \lambda_0 (a_2 - a_3) a_2 \left(r_1^2 - r_3^2 + a_3^2\right), \\ C_3 = &\quad \lambda_0 (a_2 - a_3) a_3 \left(r_1^2 - r_3^2 + a_2^2\right), \\ A = &\quad \lambda_0 a_2 a_3 (a_2 - a_3), \quad B = 0, \end{aligned}$$

where λ_0 is a constant. Then the cofactors are

$$\begin{aligned} K_1 = &\quad y\lambda_0 \left(a_2 a_3 (a_2 - a_3)x + a_2 a_3 \left(r_3^2 - r_2^2 + a_2^2 - a_3^2\right)\right), \\ K_2 = &\quad y\lambda_0 \left(a_2 a_3 (a_2 - a_3)x + (a_2 - a_3) a_2 \left(r_1^2 - r_3^2 + a_3^2\right)\right), \\ K_3 = &\quad y\lambda_0 \left(a_2 a_3 (a_2 - a_3)x + (a_2 - a_3) a_3 \left(r_1^2 - r_2^2 + a_2^2\right)\right). \end{aligned} \tag{19}$$

Thus there exists constants λ_1, λ_2 and λ_3 such that

$$K_1\lambda_1 + K_2\lambda_2 + K_3\lambda_3 = 0.$$

These constants are given by the formula (17). Consequently we have the first integral

$$F = |g_1|^{a_2 - a_3} |g_2|^{a_3 - a_1} |g_3|^{a_1 - a_2}.$$

This first integral prevents that the invariant circles are limit cycles.

It is easy to observe that the cofactors (19) can be written as

$$\begin{aligned} K_n = &\quad \sum_{j=1}^{3} \lambda_j \{g_j, g_n\} \left(\prod_{\substack{n=1 \\ n \neq j}}^{3} g_m \right) \\ = &\quad \prod_{m=1}^{3} g_m \begin{vmatrix} \{\log|g_1|, \log|g_n|\} & \{\log|g_2|, \log|g_n|\} & \{\log|g_3|, \log|g_n|\} \\ 1 & 1 & 1 \\ a_1 & a_2 & a_3 \end{vmatrix}, \end{aligned}$$

where λ_1, λ_2 and λ_3 are given by formula (17).

(iii) If $b_3 = 0$ and $a_3 = 0$, then the system (18) admits the solutions

$$C_1 = C_3 = A = B = 0,$$

and C_2 is an arbitrary constant.

In this case $K_1 = K_3 = 0$ and $K_2 = 4a_2 y C_2$. Thus differential system (15) takes the form

$$\dot{x} = -C_2 g_2 y, \quad \dot{y} = C_2 g_2 x.$$

Hence $g_2 = 0$ is a singular circle. By considering that this system admits the analytic first integral $x^2 + y^2$, then this system has no limit cycles. □

Remark 12. *The differential system* (9) *with three invariant circles is a polynomial vector field of degree 3 if and only if the constants* λ_1, λ_2 *and* λ_3 *satisfies the linear system*

$$\begin{aligned} \lambda_1+\lambda_2+\lambda_3 &= 0,\\ a_1\lambda_1+a_2\lambda_2+a_3\lambda_3 &= 0,\\ b_3\lambda_3 &= 0. \end{aligned}$$

Thus if $b_3\neq 0$ *then* $\lambda_1=\lambda_2=\lambda_3=0$. *If* $b_3=0$ *then by considering that* $a_1-a_2\neq 0$, *then* λ_j *for* $j=1,2,3$ *are given by the formula* (17).

Note that the solutions of the λ*'s are the same than in the proof of Proposition 11, and consequently the cubic vector fields coincide.*

Proposition 13. *The polynomial vector field of degree four with four different invariant circles*

$$\begin{aligned} g_1 &= (x-a_1)^2+y^2-r_1^2=0,\\ g_2 &= (x-a_2)^2+y^2-r_2^2=0,\\ g_3 &= (x-a_3)^2+y^2-r_3^2=0,\\ g_4 &= (x-a_4)^2+(y-b_4)^2-r_4^2=0,\\ \{g_1,g_2\} &= 4(a_2-a_1)y, \quad a_1\neq a_2, \end{aligned} \tag{20}$$

is the zero vector field if

$$b_4\neq 0, \quad \text{or} \quad H:=\begin{vmatrix} r_1^2 & r_2^2 & r_3^2 & r_4^2\\ 1 & 1 & 1 & 1\\ a_1 & a_2 & a_3 & a_4\\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \end{vmatrix}\neq 0,$$

or Darboux integrable if $b_4=0$ *and* $H=0$ *with the first integral* $F=\displaystyle\prod_{j=1}^{3}|g_j|^{\lambda_j}$, *where* $\lambda_1,\lambda_2,\lambda_3,\lambda_4$

(i) *are either*

$$\begin{aligned} \lambda_1&=\lambda_0\begin{vmatrix} 1 & 1 & 1\\ a_2 & a_3 & a_4\\ a_2^2 & a_3^2 & a_4^2 \end{vmatrix}, \quad \lambda_2=-\lambda_0\begin{vmatrix} 1 & 1 & 1\\ a_1 & a_3 & a_4\\ a_1^2 & a_3^2 & a_4^2 \end{vmatrix},\\ \lambda_3&=-\lambda_0\begin{vmatrix} 1 & 1 & 1\\ a_1 & a_2 & a_4\\ a_1^2 & a_2^2 & a_4^2 \end{vmatrix}, \quad \lambda_4=\lambda_0\begin{vmatrix} 1 & 1 & 1\\ a_1 & a_2 & a_3\\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix}. \end{aligned} \tag{21}$$

with

$$r_j^2=\alpha+\beta\,a_j+\gamma\,a_j^2 \quad \text{for} \quad j=1,2,3,4,$$

(ii) *or*

$$\lambda_1=-\lambda_3=A\lambda_0(r_2^2-r_4^2), \qquad \text{for} \quad \lambda_2=-\lambda_4=A\lambda_0(r_3^2-r_1^2),$$

where λ_0 *is an arbitrary constant,* $a_1=a_3$, *and* $a_2=a_4$.

Proof. We determine the solutions of the equations

$$\begin{aligned} K_3g_3\{g_1,g_2\}+K_1g_1\{g_2,g_3\}+K_2g_2\{g_3,g_1\} &= 0,\\ K_4g_4\{g_1,g_2\}+K_1g_1\{g_2,g_4\}+K_2g_2\{g_4,g_1\} &= 0, \end{aligned} \tag{22}$$

where

$$K_j=4a_2y\left(C_j+A_jx+B_jy+Lx^2+Mxy+Ny^2\right), \quad j=1,2,3,4.$$

Here we have used Proposition 8. These equations are polynomials of degree 5. By solving these equations we obtain a linear system with respect to A_j, B_j, C_j, L, M N for $j = 1, 2, 3, 4$.

$$
\begin{aligned}
b_4\,(a_2L + A_1 - A_2) = 0, \quad b_4 a_2 r_1^2 C_1 &= \ 0,\\
b_4\left(A_1 a_2 r_1^2 - r_1^2 C_1 + C_2(r_2^2 - a_2^2)\right) &= \ 0,\\
b_4\left(r_1^2 A_1 + A_2(a_2^2 - r_2^2) + a_2 C_1 - a_2 r_1^2 L\right) &= \ 0,\\
b_4\left(L\left(r_1^2 + a_2^2 - r_2^2\right) + a_2 A_1 - 2a_2 A_2 + C_2 - C_1\right) &= \ 0,\\
\vdots \qquad\qquad \vdots \qquad\qquad & \ \vdots
\end{aligned}
\tag{23}
$$

There are 25 equations. By using the algebraic manipulator we obtain that if $b_4 \neq 0$ or $H \neq 0$, then we get that the unique solution of the linear system (23) with respect to the 15 unknowns A_j, B_j, C_j, L, M N for $j = 1, 2, 3, 4$, are the trivial solutions

$$A_j = B_j = C_j = L = M = N = 0, \quad \text{for} \quad j = 1, 2, 3, 4.$$

Thus $K_1 = K_2 = K_3 = K_4 = 0$. Consequently the vector field is the zero vector field. Hence the proposition is proved if $b_4 \neq 0$ or $H \neq 0$.

If $b_4 = 0$ and $H = 0$, then system (23) is formed for 20 equations which admits non-trivial solutions. Indeed if $H = 0$ then $r_j^2 = \alpha + \beta a_j + \gamma a_j^2$ for $j = 1, 2, 3, 4$, or $\operatorname{rank}(\Lambda) = 2$, where

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \end{pmatrix}.$$

In the first case under the assumption that $\operatorname{rank}(\Lambda) \neq 2$ we obtain that the cofactors are

$$
\begin{aligned}
K_1 = \ & \frac{yN}{\tilde{\gamma}}\Big((\tilde{\gamma}+4)x^2 + \tilde{\gamma}y^2 + 2x\,(\tilde{\gamma}(a_2+a_3+a_4) + 2\beta)\\
& + \beta\tilde{\gamma}(a_2+a_3+a_4) + \tilde{\gamma}^2(a_2a_3 + a_2a_4 + a_3a_4) + \beta^2\Big)\\
= \ & y\left(\Psi_2(x,y) + \Psi_1(x,y)a_1 + \Psi_0 a_1^2\right),\\
K_2 = \ & \frac{yN}{\tilde{\gamma}}\Big(((\tilde{\gamma}+4)x^2 + \tilde{\gamma}y^2 + 2x\,(\tilde{\gamma}(a_1+a_3+a_4) + 2\beta)\\
& + \beta\tilde{\gamma}(a_1+a_3+a_4) + \tilde{\gamma}^2(a_1a_3 + a_1a_4 + a_3a_4) + \beta^2\Big)\\
= \ & y\left(\Psi_2(x,y) + \Psi_1(x,y)a_1 + \Psi_0 a_2^2\right),\\
K_3 = \ & \frac{yN}{\tilde{\gamma}}\Big((\tilde{\gamma}+4)x^2 + \tilde{\gamma}y^2 + 2x\,(\tilde{\gamma}(a_1+a_2+a_4) + 2\beta)\\
& + \beta\tilde{\gamma}(a_1+a_2+a_4) + \tilde{\gamma}^2(a_1a_2 + a_1a_4 + a_2a_4) + \beta^2\Big)\\
= \ & y\left(\Psi_2(x,y) + \Psi_1(x,y)a_3 + \Psi_0 a_3^2\right),\\
K_4 = \ & \frac{yN}{\tilde{\gamma}}\Big((\tilde{\gamma}+4)x^2 + \tilde{\gamma}y^2 + 2x\,(\gamma(a_1+a_2+a_3) + 2\beta)\\
& + \beta\tilde{\gamma}(a_1+a_3+a_4) + \tilde{\gamma}^2(a_1a_3 + a_1a_2 + a_3a_2) + \beta^2\Big)\\
= \ & y\left(\Psi_2(x,y) + \Psi_1(x,y)a_4 + \Psi_0 a_4^2\right),
\end{aligned}
$$

where $\tilde{\gamma}=\gamma-1$, Ψ_2, Ψ_1 and Ψ_0 are convenient polynomials. It is easy to show that the equation $\lambda_1K_1+\lambda_2K_2+\lambda_3K_3+\lambda_4K_4=0$, holds if and only if

$$\sum_{j=1}^{4}\lambda_j=0,\quad \sum_{j=1}^{4}a_j\lambda_j=0,\quad \sum_{j=1}^{4}a_j^2\lambda_j=0,$$

thus λ_j for $j=1,2,3,4$ are determined by the formula (21). Consequently by Theorem 2 there exists the first integral $F=\prod_{j=1}^{4}|g_j|^{\lambda_j}$. Therefore the proposition is proved when $b_4=0$ and $H=0$ under the conditions $r_j^2=\alpha+\beta a_j+\gamma a_j^2$ for $j=1,2,3,4$.

We observe that when $\tilde{\gamma}=0$, we obtain $K_1=K_2=K_3=K_4$. Consequently from (3) it follows that the curve $K_1=0$ is a set of the critical points of the vector field.

In the second case (i.e. $\mathrm{rank}(\Lambda)=2$) without loss the generality we suppose that $a_1=a_3=0$ and $a_2=a_4$, we have that the solutions of system (22) (i.e. $(a_2-a_1)(K_3g_3-K_1g_1)=0,\qquad (a_2-a_1)(K_4g_4-K_2g_2)=0$) are

$$\begin{aligned}
A_1=&\quad A_3=B_1=B_2=B_3=B_4=M_1=0,\\
C_1=&\quad A_4r_3^2,\quad C_2=-A_4(a_4^2-r_4^2),\\
C_3=&\quad A_4r_1^2,\quad C_4=-A_4(a_2^2-r_2^2),\\
L_1=&\quad N_1=L=-A_4,\quad A_2=A_4.
\end{aligned}$$

Therefore the cofactors are

$$\begin{aligned}
K_1=&\quad 4a_2y\left(C_1+L(x^2+y^2)\right)=-4a_2yA_4g_3,\\
K_3=&\quad 4a_2y\left(C_3+L(x^2+y^2)\right)=-4a_2yA_4g_1,\\
K_2=&\quad 4a_2y\left(C_2+2a_4A_4x+L(x^2+y^2)\right)=-4a_2A_4yg_4,\\
K_4=&\quad 4a_2y\left(C_4+2a_4A_4x+L(x^2+y^2)\right)=-4a_2A_4yg_2.
\end{aligned}$$

We consider the equation $\lambda_1K_1+\lambda_2K_2+\lambda_3K_3+\lambda_4K_4=0$, and we obtain

$$\lambda_1C_1+\lambda_2C_2+\lambda_3C_3+\lambda_4C_4=0,\quad \lambda_1+\lambda_2+\lambda_3+\lambda_4=0,\quad A(\lambda_2+\lambda_4)=0$$

Hence

$$\begin{aligned}
\lambda_1=&\quad \lambda_0(C_2-C_4)=-\lambda_3=A\lambda_0(r_2^2-r_4^2),\\
\lambda_2=&\quad \lambda_0(C_3-C_1)=-\lambda_4=A\lambda_0(r_3^2-r_1^2),
\end{aligned}\tag{24}$$

where λ_0 is an arbitrary constants. Again, by Theorem 2 there exist the first integral $F=\prod_{j=1}^{4}|g_j|^{\lambda_j}$. Hence the proposition is proved when $b_4=H=0$ under the condition $\mathrm{rank}(\Lambda)=2$. In short the quartic system with four invariant circles is Darboux integrable. Thus the proposition is proved. □

From the proof of Theorem 2 we get that the cofactors K_n of the circles $g_n = 0$ when $r_j^2 = \alpha + \beta a_j + \gamma a_j^2$ for $j = 1, 2, 3, 4.$ are

$$K_n = \sum_{j=1}^{4} \lambda_j \{g_j, g_n\} \prod_{\substack{m=1 \\ m \neq j}}^{4} g_m = \lambda_0 g \cdot$$

$$\begin{vmatrix} \{\log|g_1|, \log|g_n|\} & \{\log|g_2|, \log|g_n|\} & \{\log|g_3|, \log|g_n|\} & \{\log|g_4|, \log|g_n|\} \\ 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \end{vmatrix}$$

for $n = 1, 2, 3, 4$, where $g = \prod_{m=1}^{S} g_m$, λ_1, λ_2, λ_3 and λ_4 are given by the formula (21) and (24).

If $\mathrm{rank}(\Lambda) = 2$ (assuming that $a_1 = a_3$ and $a_2 = a_4$), then the cofactors K_n are

$$K_n = \sum_{j=1}^{4} \lambda_j \{g_j, g_n\} \prod_{\substack{m=1 \\ m \neq j}}^{4} g_m = \lambda_0 g \cdot$$

$$\begin{vmatrix} \{\log|g_1|, \log|g_n|\} & \{\log|g_2|, \log|g_n|\} & \{\log|g_3|, \log|g_n|\} & \{\log|g_4|, \log|g_n|\} \\ 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_1 & a_2 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 \end{vmatrix}$$

for $n = 1, 2, 3, 4$, From here we observe that $K_1 g_1 = K_3 g_3$ and $K_2 g_2 = K_4 g_4$.

The case when $r_j^2 = \alpha + \beta a_j + \gamma a_j^2$ for $j = 1, 2, 3, 4$, and $\mathrm{rank}(\Lambda) = 2$, implies that $g_1 = g_3$ and $g_2 = g_4$.

Remark 14. *The polynomial vector field* (9) *with the four invariant circles* (20) *is a polynomial vector field of degree four if and only if the constants* $\lambda_1, \lambda_2, \lambda_3$ *and* λ_4 *are such that*

$$\begin{aligned} \sum_{j=1}^{4} a_j^n \lambda_j &= 0, \quad \textit{for} \quad n = 0, 1, 2, \\ b_3\lambda_3 + b_4\lambda_4 &= 0, \\ a_3 b_3 \lambda_3 + a_4 b_4 \lambda_4 &= 0, \\ b_3 b_4 (\lambda_3 + \lambda_4) &= 0, \\ \sum_{j=1}^{4} r_j^2 \lambda_j - b_3^2 \lambda_3 - b_4^2 \lambda_4 &= 0. \end{aligned} \tag{25}$$

Thus if $b_3^2+b_4^2\neq 0$, *then* $\lambda_1=\lambda_2=\lambda_3=\lambda_4=0$. *Thus the vector field is the zero vector field. If* $b_3=b_4=0$ *then system* (25) *becomes*

$$\begin{aligned}\sum_{j=1}^{4} a_j^n\lambda_j &= 0, \quad \textit{for} \quad n=0,1,2,\\ \sum_{j=1}^{4} r_j^2\lambda_j &= 0.\end{aligned} \tag{26}$$

This system admits nontrivial solutions if and only if the matrix

$$\Omega=\begin{pmatrix} r_1^2 & r_2^2 & r_3^2 & r_4^2\\ 1 & 1 & 1 & 1\\ a_1 & a_2 & a_3 & a_4\\ a_1^2 & a_2^2 & a_3^2 & a_4^2\end{pmatrix}$$

has rank smaller than 4. By considering that $a_1\neq a_2$ *then rank*$(\Omega)=3$ *if one of the following three conditions hold:*

either $r_j^2=\alpha+\beta a_j+\gamma a_j^2 \quad \textit{for} \quad j=1,2,3,4,$

or $a_1=a_3, \quad a_2=a_4,$

or $a_1=a_4, \quad a_2=a_3$. *In the first case we obtain that the constants* λ*'s are given in* (21).

In the second case it is easy to show that the solutions of the linear system (26) *are* $\lambda_1=-\lambda_3=\lambda_0(r_2^2-r_4^2), \quad \lambda_2=-\lambda_4=\lambda_0(r_3^2-r_1^2)$, *hence we have the first integral* $F=\prod_{j=1}^{4}|g_j|^{\lambda_j}$.

The third case follows in a similar way to the second case.

Note that the solutions of the λ's are the same than in the proof of Proposition 13, therefore the quartic vector fields coincide.

Now we shall extend the results of Proposition 11 and 13 to the configuration of S circles with $S\geq 4$.

Proposition 15. *We consider the configuration of* $S\geq 4$ *circles of the form*

$$g_j=(x-a_j)^2+y^2-r_j^2=0, \quad \textit{for} \quad j=1,\dots,S, \tag{27}$$

with $\{g_1,g_2\}=4(a_2-a_1)y$, $a_2-a_1\neq 0$, *and the matrix*

$$\tilde{\Lambda}=\begin{pmatrix} 1 & 1 & \dots & \dots & 1\\ a_1 & a_2 & \dots & \dots & a_S\\ a_1^2 & a_2^2 & \dots & \dots & a_S^2\\ \vdots & \vdots & \dots & \dots & \vdots\\ a_1^{S-2} & a_2^{S-2} & \dots & \dots & a_S^{S-2}\\ r_1^2 & r_2^2 & \dots & \dots & r_S^2\\ a_1r_1^2 & a_2r_2^2 & \dots & \dots & a_Sr_S^2\\ \vdots & \vdots & \dots & \dots & \vdots\\ a_1^l r_1^{2k} & a_2^l r_2^{2k} & \dots & \dots & a_S^l r_S^{2k}\end{pmatrix},$$

where after the row $a_1^{S-2},\dots,a_S^{S-2}$ *come all the rows of the form* $a_1^l r_1^{2k}, a_2^l r_2^{2k},\dots,a_S^l r_S^{2k}$ *with* $l+2k=2,3,\dots,S-2$, *thus for* $l+2k=2$ *we have the row* $r_1^2,\dots,r_S^2$, *for* $l+2k=3$ *we have the row* $a_1r_1^2,\dots,a_Sr_S^2$, *for* $l+2k=3$ *we have two rows,....*

The vector field (3) *under the conditions* (2) *is Darboux integrable and it is not a zero vector field if and only if the matrix* $\tilde{\Lambda}$ *satisfies rank*$(\tilde{\Lambda})<S$.

Proof. We suppose that a planar polynomial vector field with the S invariant circles (27) is Darboux integrable and it is not the zero vector field, then by Theorem 2 this polynomial differential system (1) becomes system (9). For the circles (27) system (9) can be written as

$$\begin{aligned}\dot{x} =&\quad -y(x^2+y^2)^{S-1}\sum_{j=1}^{S}\lambda_j-(x^2+y^2)^{S-2}R_1(x,y)+\ldots=P(x,y),\\ \dot{y} =&\quad x(x^2+y^2)^{S-1}\sum_{j=1}^{S}\lambda_j-(x^2+y^2)^{S-2}R_2(x,y)+\ldots=Q(x,y),\end{aligned}\tag{28}$$

where

$$\begin{aligned}R_1 =&\quad y\sum_{j=1}^{S}\left(-2xa_j+a_j^2-r_j^2\right)\sum_{m=1}^{S}\lambda_m+y\sum_{j=1}^{S}\lambda_j\left(2xa_j-a_j^2+r_j^2\right),\\ R_2 =&\quad \sum_{j=1}^{S}\left(-2xa_j+a_j^2-r_j^2\right)\sum_{m=1}^{S}(y-a_m)\lambda_m\\ &\sum_{j=1}^{S}(y-a_j)\lambda_j\left(2xa_j-a_j^2+r_j^2\right)-(x^2+y^2)\sum_{j=1}^{S}a_j\lambda_j.\end{aligned}\tag{29}$$

Since we want that $\max\left(\deg P(x,y),\deg Q(x,y)\right)=S$ we obtain the following linear system in the unknowns $\lambda_1,\dots,\lambda_S$

$$\sum_{j=1}^{S}\tau_j^{(n)}(a_1,\dots,a_S,r_1^2,\dots,r_S^2)\lambda_j=0,\tag{30}$$

where $\tau_j^{(n)}(a_1,\dots,a_S,r_1^2,\dots,r_S^2)$ are homogenous polynomials of degree n on the variables $a_1,\dots,a_S,r_1^2,\dots,r_S^2$ where $j=1,\dots,S$ and $n=0,\dots,S_1<(3S+2)(S-1)/2$. Some of the equations (30) are

$$\begin{aligned}\sum_{j=1}^{S}\lambda_j =&\quad 0,\quad \sum_{j=1}^{S}a_j\lambda_j=0,\quad \sum_{j=1}^{S}a_j^2\lambda_j=0,\\ \sum_{j=1}^{S}r_j^2\lambda_j =&\quad 0,\quad \sum_{j=1}^{S}a_jr_j^2\lambda_j=0,\end{aligned}$$

obtained directly from (29) and (30). After some computations we can show that (30) can be written as follows

$$\sum_{j=1}^{S} a_j^n \lambda_j = \quad 0, \quad \text{for} \quad n = 0, \ldots, S-2,$$

$$\sum_{j=1}^{S} a_j^l r_j^{2k} \lambda_j = \quad 0, \quad \text{for} \quad l + 2k = 2, \ldots, S-2,\ k > 0,$$

or equivalently

$$\Lambda\, \lambda = 0, \quad \lambda = (\lambda_1, \ldots, \lambda_S)^T .$$

Thus if $\operatorname{rank}(\tilde{\Lambda}) = S$, then $\lambda_j = 0$ for $j = 1, \ldots, S$. Since by assumption system (28) is a non-zero vector field, we must have that $\operatorname{rank}(\tilde{\Lambda}) < S$.

We prove the reciprocal. From (14) with $b_j = 0$ we deduce the equations

$$K_j g_j (a_2 - a_1) + K_1 g_1 (a_1 - a_j) + K_1 g_1 (a_j - a_2) = 0, \tag{31}$$

for $j = 3, \ldots, S$.

First we study the case when

$$\prod_{1 \le j < k \le S} (a_j - a_k) \neq 0, \tag{32}$$

and the radii of the circles satisfy

$$r_j^2 = \alpha + \beta a_j + \gamma a_j^2, \quad \text{for} \quad j = 1, \ldots, S. \tag{33}$$

The solutions of equations (31) are

$$K_j = g \sum_{n=1}^{S} \lambda_n \{\log|g_n|, \log|g_j|\} =$$

$$g \begin{vmatrix} \{\log|g_1|, \log|g_j|\} & \ldots & 0 & \ldots & \{\log|g_S|, \log|g_j|\} \\ 1 & \ldots & 1 & \ldots & 1 \\ a_1 & \ldots & a_j & \ldots & a_S \\ a_1^2 & \ldots & a_j^2 & \ldots & a_S^2 \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ a_1^{S-3} & \ldots & a_j^{S-3} & \ldots & a_S^{S-3} \\ a_1^{S-2} & \ldots & a_j^{S-2} & \ldots & a_S^{S-2} \end{vmatrix} . \tag{34}$$

where $g = \prod_{n=1}^{S} g_n$.

It is easy to show that these cofactors are polynomial of degree $S-1$ and that they can be written as in (12).

In view of the fact that the matrix with elements $(\{\log|g_n|, \log|g_j|\})$ is skewsymmetric, then the equation $\sum_{j=1}^{S} \lambda_j K_j = 0$, holds, thus the first integral $F = \prod_{j=1}^{S} |g_j|^{\lambda_j}$ exists.

Analogously we can study the case when some of the circles have the same center with different radii. Assume that

$$\begin{aligned} a_1 = &\quad a_3 = \ldots = a_{k_1}, \\ a_2 = &\quad a_{k_1+1} = \ldots = a_{k_2}, \\ &\quad \vdots \qquad\quad \vdots, \\ a_{k_j+1} = &\quad a_{k_j+2} = \ldots = a_S. \end{aligned}$$

It is possible to show that cofactors of these circles which are solutions of (31) are such that

$$\begin{aligned} K_n g_n = &\quad K_1 g_1 \quad \text{for} \quad n = 3, \ldots, k_1, \\ K_n g_n = &\quad K_2 g_2 \quad \text{for} \quad n = k_1 + 1, \ldots, k_2, \\ \vdots \quad &\quad \vdots \quad \vdots \quad \vdots \\ K_n g_n = &\quad K_S g_S \quad \text{for} \quad n = k_j + 1, \ldots, S - 1. \end{aligned}$$

It is possible to show that the cofactors are

$$K_j = g \sum_{n=1}^{S} \lambda_n \{\log|g_n|, \log|g_j|\} =$$

$$g \begin{vmatrix} \{\log|g_1|, \log|g_j|\} & \ldots & 0 & \ldots & \{\log|g_S|, \log|g_j|\} \\ 1 & \ldots & 1 & \ldots & 1 \\ a_1 & \ldots & a_j & \ldots & a_S \\ a_1^2 & \ldots & a_j^2 & \ldots & a_S^2 \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ a_1^N & \ldots & a_j^N & \ldots & a_S^N \\ r_1^2 & \ldots & r_j^2 & \ldots & r_S^2 \\ a_1 r_1^2 & \ldots & a_j r_j^2 & \ldots & a_S r_S^2 \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ a_1^l r_1^{2k} & \ldots & a_j^l r_j^{2k} & \ldots & a_S^l r_S^{2k} \end{vmatrix},$$

where $l + 2k = S - 2$, and $N < S - 2$. These cofactors are polynomial of degree at most $S - 1$, and the relations (12) hold.

By considering that the matrix with elements $H := (\{\log|g_n|, \log|g_j|\})$ is skewsymmetric then the relation $\sum_{j=1}^{S} \lambda_j K_j = 0$ holds, thus there exists the first integral $F = \prod_{j=1}^{S} |g_j|^{\lambda_j}$. □

Proposition 16. *The polynomial vector field of degree S with the S invariant circles* (27) *for which* (32) *and* (33) *hold is the vector field*

$$\begin{aligned}\mathcal{X} = &\ g\sum_{n=1}^{S}\lambda_n\{\log|g_n|,*\}\\ = &\ g\begin{vmatrix} \{\log|g_1|,*\} & \dots & \{\log|g_j|,*\} & \dots & \{\log|g_S|,*\}\\ 1 & \dots & 1 & \dots & 1\\ a_1 & \dots & a_j & \dots & a_S\\ a_1^2 & \dots & a_j^2 & \dots & a_S^2\\ \vdots & \dots & \vdots & \dots & \vdots\\ a_1^{S-3} & \dots & a_j^{S-3} & \dots & a_S^{S-3}\\ a_1^{S-2} & \dots & a_j^{S-2} & \dots & a_S^{S-2}\end{vmatrix},\end{aligned}$$

where $\{f,*\} = \dfrac{\partial f}{\partial x}\dfrac{\partial}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial}{\partial x}$, *with* λ_n *a constant for* $n = 1,\dots,S$. *The vector field* $\mathcal{X}$ *is Darboux integrable.*

Proof. The vector field with S invariant circles can be written in the form $\mathcal{X} = (P,Q)$ where P and Q are given by the formula (4) and the cofactors of the given circles are given by the formula (34). Thus we obtain that

$$\begin{aligned}\mathcal{X}(*) = &\ \frac{\displaystyle\sum_{j=1}^{S}\lambda_j\left(\prod_{\substack{m=1\\m\neq j}}^{S} g_m\right)\{g_1,\,g_j\}}{\{g_1,g_2\}}\{*,g_2\} + \frac{\displaystyle\sum_{j=1}^{S}\lambda_j\left(\prod_{\substack{m=1\\m\neq j}}^{S} g_m\right)\{g_2,\,g_j\}}{\{g_1,g_2\}}\{g_1,*\}\\ = &\ \frac{\displaystyle\sum_{j=1}^{S}\lambda_j\left(\prod_{\substack{m=1\\m\neq j}}^{S} g_m\right)}{\{g_1,g_2\}}\left(\{g_1,\,g_j\}\{*,g_2\} + \{g_2,\,g_j\}\{g_1,*\}\right)\\ = &\ \sum_{j=1}^{S}\lambda_j\left(\prod_{\substack{m=1\\m\neq j}}^{S} g_m\right)\{g_j,\,*\},\end{aligned}$$

where λ_j for $j = 1,\dots,S$ are constants. Thus in view of Corollary 7, the vector field is Darboux integrable. Hence

$$\mathcal{X}(*) = g\begin{vmatrix} \{\log|g_1|,*\} & \dots & \{\log|g_j|,*\} & \dots & \{\log|g_S|,*\}\\ 1 & \dots & 1 & \dots & 1\\ a_1 & \dots & a_j & \dots & a_S\\ a_1^2 & \dots & a_j^2 & \dots & a_S^2\\ \vdots & \dots & \vdots & \dots & \vdots\\ a_1^{S-3} & \dots & a_j^{S-3} & \dots & a_S^{S-3}\\ a_1^{S-2} & \dots & a_j^{S-2} & \dots & a_S^{S-2}\end{vmatrix},$$

where $g = \displaystyle\prod_{n=1}^{S} g_n$.

First we shall see that the vector field $\mathcal{X}$ has degree S and has the invariant circles (27). Indeed,

$$\mathcal{X} = 2(x^2+y^2)^{S-1}\begin{vmatrix} 1 & \dots & 1 & \dots & 1 \\ 1 & \dots & 1 & \dots & 1 \\ a_1 & \dots & a_j & \dots & a_S \\ a_1^2 & \dots & a_j^2 & \dots & a_S^2 \\ \vdots & \dots & \vdots & \dots & \vdots \\ a_1^{S-3} & \dots & a_j^{S-3} & \dots & a_S^{S-3} \\ a_1^{S-2} & \dots & a_j^{S-2} & \dots & a_S^{S-2} \end{vmatrix}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$+2x(x^2+y^2)^{S-2}\begin{vmatrix} a_1 & \dots & a_j & \dots & a_S \\ 1 & \dots & 1 & \dots & 1 \\ a_1 & \dots & a_j & \dots & a_S \\ a_1^2 & \dots & a_j^2 & \dots & a_S^2 \\ \vdots & \dots & \vdots & \dots & \vdots \\ a_1^{S-3} & \dots & a_j^{S-3} & \dots & a_S^{S-3} \\ a_1^{S-2} & \dots & a_j^{S-2} & \dots & a_S^{S-2} \end{vmatrix}\frac{\partial}{\partial y} + \dots$$

after some calculations we obtain that the vector field $\mathcal{X}$ is a polynomial vector field of degree S. On the other hand from the relations

$$\mathcal{X}(g_j) = \prod_{m=1}^{S} g_m \begin{vmatrix} \{\log|g_1|, g_j|\} & \dots & 0 & \dots & \{\log|g_S|, g_j\} \\ 1 & \dots & 1 & \dots & 1 \\ a_1 & \dots & a_j & \dots & a_S \\ a_1^2 & \dots & a_j^2 & \dots & a_S^2 \\ \vdots & \dots & \vdots & \dots & \vdots \\ a_1^{S-3} & \dots & a_j^{S-3} & \dots & a_S^{S-3} \\ a_1^{S-2} & \dots & a_j^{S-2} & \dots & a_S^{S-2} \end{vmatrix} = g_j K_j,$$

for $j = 1, \dots, S$. Consequently $g_j = 0$ for $j = 1, \dots, S$ are invariant algebraic curves of the polynomial vector field of degree S. □

Proposition 17. *The polynomial vector field of degree S with the S invariant circles*

$$\begin{aligned} g_j &= (x-a_1)^2 + y^2 - r_j^2 = 0 \quad \textit{for} \quad j = 1, 3, \dots, k_1, \\ g_j &= (x-a_2)^2 + y^2 - r_j^2 = 0 \quad \textit{for} \quad j = 2, k_1+1, \dots, k_2, \\ &\vdots \qquad\qquad \vdots \\ g_j &= (x-a_S)^2 + y^2 - r_j^2 = 0 \quad \textit{for} \quad j = k_j+1, \dots, S, \end{aligned}$$

having different radii for the circles with the same center, can be written as

$$\begin{aligned}\mathcal{X} = & \; g\sum_{n=1}^{S}\lambda_n\{\log|g_n|,*\}\\ = & \; g\begin{vmatrix} \{\log|g_1|,*\} & \dots & 0 & \dots & \{\log|g_S|,*\}\\ 1 & \dots & 1 & \dots & 1\\ a_1 & \dots & a_j & \dots & a_S\\ a_1^2 & \dots & a_j^2 & \dots & a_S^2\\ \vdots & \dots & \vdots & \dots & \vdots\\ a_1^N & \dots & a_j^N & \dots & a_S^N\\ r_1^2 & \dots & r_j^2 & \dots & r_S^2\\ a_1r_1^2 & \dots & a_jr_j^2 & \dots & a_Sr_S^2\\ \vdots & \dots & \vdots & \dots & \vdots\\ a_1^lr_1^{2k} & \dots & a_j^lr_j^{2k} & \dots & a_S^lr_S^{2k}\end{vmatrix},\end{aligned}$$

where $l+2k=S-2$, and $N<S-2$, and λ_n is a constant for $n=1,\dots,S$. The vector field $\mathcal{X}$ is Darboux integrable.

Proof. Its proof is analogous to the proof of the previous proposition. □

Proposition 18. *The unique polynomial vector field with the S invariant circles*

$$\begin{aligned} g_1 = & \; (x-a_1)^2+y^2-r_1^2=0,\\ g_2 = & \; (x-a_2)^2+y^2-r_2^2=0,\\ g_j = & \; (x-a_j)^2+(y-b_j)^2-r_S^2=0, \quad \textit{for} \quad j=3,\dots,S,\end{aligned} \tag{35}$$

with $\sum_{j=3}^{S}b_j^2\neq 0$, is the zero vector field.

Proof. The integrability of the polynomial vector field of degree S with S invariant circles

$$g_j=(x-a_j)^2+(y-b_j)^2-r_j^2=0 \quad \text{for} \quad j=1,\dots,S, \tag{36}$$

follows from differential system (1). Indeed, by Proposition 6 and Theorem 5 we know that any polynomial differential system having the circles (36) as invariant algebraic curves can be written as system (1), i.e.

$$\begin{aligned}\dot{x} = & \; -\lambda_{S+1}\prod_{m=1}^{S}g_m-2\sum_{j=1}^{S}\lambda_j(y-b_j)\left(\prod_{\substack{m=1\\m\neq j}}^{S}g_m\right)=P(x,y),\\ \dot{y} = & \; \lambda_{S+2}\prod_{m=1}^{S}g_m+2\sum_{j=1}^{S}\lambda_j(x-a_j)\left(\prod_{\substack{m=1\\m\neq j}}^{S}g_m\right)=Q(x,y),\end{aligned} \tag{37}$$

where $\lambda_1, \ldots, \lambda_{S+2}$ are arbitrary polynomials. Thus if $\max(\deg(P), \deg(Q)) = S$ then $\lambda_{S+1} = \lambda_{S+2} = 0$, we obtain the system

$$\dot{x} = -2\sum_{j=1}^{S} \lambda_j (y - b_j) \left(\prod_{\substack{m=1 \\ m \neq j}}^{S} g_m \right), \quad \dot{y} = 2\sum_{j=1}^{S} \lambda_j (x - a_j) \left(\prod_{\substack{m=1 \\ m \neq j}}^{S} g_m \right),$$

where λ_j for $j = 1, \ldots, S$ are polynomials of degree κ_j, i.e.

$$\lambda_j = A_j x + B_j y + C_j + M_j x^2 + N_j xy + L_j y^2 + \ldots + \mu_j y^{\kappa_j}.$$

By requiring that this differential system is a polynomial vector field of degree S we obtain that we must eliminate at most $\frac{1}{2}(S + \kappa - 1)(3S + \kappa + 2)$ coefficients where $\kappa = \max(\kappa_1, \ldots, \kappa_S)$. Thus we have the following relations

$$\begin{aligned}
&\sum_{j=1}^{S} \tau_{3,j}^{(n_1)} C_j = 0, \quad \sum_{j=1}^{S} \tau_{1,j}^{(n_2)} A_j = 0, \quad \sum_{j=1}^{S} \tau_{2,j}^{(n_3)} B_j = 0, \\
&\sum_{j=1}^{S} \tau_{2,j}^{(n_4)} N_j = 0, \quad \sum_{j=1}^{S} \tau_{2,j}^{(n_5)} M_j = 0, \quad \sum_{j=1}^{S} \tau_{2,j}^{(n_6)} L_j = 0, \\
&\qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{aligned} \tag{38}$$

where $\tau_{m,j}^{(n_k)} = \tau_{m,j}^{(n_k)}(a_1, \ldots, a_S, b_1, \ldots, b_S, r_1^2, \ldots, r_S^2)$ are homogenous polynomials of degree n_k on the variables $a_1, \ldots, a_S, b_1, \ldots, b_S, r_1^2, \ldots, r_S^2$ where $j = 1, \ldots, S$. After some computations we prove that (38) is equivalent to the linear system in the unknowns A_j, $B_j, C_j, M_j, N_j, L_j, \ldots$ for $j = 1, \ldots, S$

$$\begin{aligned}
\sum_{j=1}^{S} a_j^n C_j &= 0, \quad \text{for} \quad n = 0, \ldots, S-2, \\
\sum_{j=1}^{S} a_j^l r_j^{2k} C_j &= 0, \quad \text{for} \quad l + 2k = 2, \ldots, S-2 \\
\sum_{j=1}^{S} a_j^n A_j &= 0, \quad \text{for} \quad n = 0, \ldots, S-1, \\
\sum_{j=1}^{S} a_j^l r_j^{2k} A_j &= 0, \quad \text{for} \quad l + 2k = 2, \ldots, S-1, \\
\sum_{j=1}^{S} a_j^n B_j &= 0, \quad \text{for} \quad n = 0, \ldots, S-1, \\
\sum_{j=1}^{S} a_j^l r_j^{2k} B_j &= 0, \quad \text{for} \quad l + 2k = 2, \ldots, S-1, \\
\sum_{j=1}^{S} a_j^n M_j &= 0, \quad \text{for} \quad n = 0, \ldots, S,
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{S} a_j^l r_j^{2k} M_j &= 0, \quad \text{for} \quad l+2k=2,\dots,S,\\
\sum_{j=1}^{S} a_j^n N_j &= 0, \quad \text{for} \quad n=0,\dots,S,\\
\sum_{j=1}^{S} a_j^l r_j^{2k} N_j &= 0, \quad \text{for} \quad l+2k=2,\dots,S,\\
\sum_{j=1}^{S} a^n L_j &= 0, \quad \text{for} \quad n=0,\dots,S,\\
\sum_{j=1}^{S} a_j^l r_j^{2k} L_j &= 0, \quad \text{for} \quad l+2k=2,\dots,S,\\
\sum_{j=1}^{S} b_j C_j &= 0,\\
\vdots \qquad & \vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

Consequently we have that $A_j = B_j = M_j = N_j = L_j = C_j = \ldots = 0$, for $j = 1,\dots,S$, if $\sum_{j=1}^{S} b_j^2 \neq 0$. Hence we obtain that $\lambda_j = 0$ and consequently the vector field is a zero vector field. □

Proof of Theorem 3. The integrability for the case when the circles are given by the formula $g_j = (x-a_j)^2 + y^2 - r_j^2 = 0$ follows from Propositions 16 and 17. For the circles (35) follows from Proposition 18. □

5. Proof of Theorem 4

The next result is due to Christopher [2].

Theorem 19. *Let $g = 0$ be a real non-singular algebraic curve of degree n, and h a first degree polynomial, chosen so that the real straight line $h = 0$ lies outside all ovals of $g = 0$. Choose the real numbers a and b so that $ah_x + bh_y \neq 0$, then the polynomial vector field of degree n,*

$$\dot{x} = ag - hg_y, \qquad \dot{y} = bg + hg_x,$$

has all the ovals of $g = 0$ as hyperbolic limit cycles. Furthermore this vector field has no other limit cycles.

From Theorem 3 we have that the polynomial vector field of degree S with S invariant circles does not admits limit cycles. We claim that a polynomial vector field of degree S can have at most $S-1$ algebraic limit cycles given by circles. We denote by $A(S)$ the maximum number of algebraic limit cycles given by circles which admits a polynomial vector fields of degree S.

Corollary 20. *A(2)=1.*

Proof. Indeed, by Theorem 19 taking $a = 0$, $b = 1$, $g = x^2 + y^2 - 1$ and $h = y + 2$, it follows that the polynomial differential system

$$\dot{x} = 2(y + 2)y, \quad \dot{y} = x^2 + y^2 - 1 - 2(y + 2)x.$$

of degree $m = 2$ has the circle $x^2 + y^2 - 1 = 0$ as an algebraic limit cycle, which is the unique limit cycle of this system. Thus we have that $A(2) \geq 1$. Now we prove that $A(2) = 1$. By Proposition 10 the quadratic planar vector fields with two invariant circles are rational integrable, consequently the quadratic system has no limit cycles. So the claim is proved. □

Now we prove that $A(S) \geq S - 1$, $S \geq 2$.

Proposition 21. *Consider the polynomial differential system*

$$\begin{aligned} \dot{x} &= \Big(F_0(x, y) - F_a(x, y)\Big)\, y = P(x, y), \\ \dot{y} &= -\Big(F_0(x, y) - F_a(x, y)\Big)\, x + aF_0(x, y) = Q(x, y), \end{aligned} \tag{39}$$

where

$$F_a(x, y) = (x + y - a) \prod_{j=1}^{S} ((x - a)^2 + y^2 - r_j^2), \quad F_0(x, y) = F_a(x, y)|_{a=0},$$

of degree $m = 2S + 1$ if $a \neq 0$, then system (39) has only three equilibrium points: $(0, 0)$ and $(a, 0)$ which are foci, and $(a/2, 0)$ which is a saddle. Moreover the circles $x^2 + y^2 - r_j^2 = 0$ and $(x - a)^2 + y^2 - r_j^2 = 0$ for $j = 1, \ldots, S$ are limit cycles of the system if $0 < r_1 < r_2 < \ldots, < r_S < a/2$.

Proof. First we claim that system (39) has the following three singular points in $\mathbb{R}^2$: $(0, 0), (a/2, 0), (a, 0)$. Now we prove the claim. First we show that there are no singular points (x_0, y_0) on the curve $F_0(x, y) - F_a(x, y) = 0$, i.e. $F_0(x_0, y_0) - F_a(x_0, y_0) = 0$; otherwise from (39) we should have $F_0(x_0, y_0) = 0$, and consequently $F_a(x_0, y_0) = 0$, and this a contradiction. Thus the singular points of system (39) are of the type $(x_0, 0)$ where x_0 is a zero of the function

$$G(x) = x(x - a) \left(\prod_{j=1}^{S} ((x - a)^2 - r_j^2) - \prod_{j=1}^{S} (x^2 - r_j^2) \right).$$

The function $R(x) := G(x + a/2)$

$$R(x) = (x - a/2)(x + a/2) \left(\prod_{j=1}^{S} ((x - a/2)^2 - r_j^2) - \prod_{j=1}^{S} ((x + a/2)^2 - r_j^2) \right),$$

is such that $G(-x) = -G(x)$. This function can be written as

$$R(x) = (x - a/2)(x + a/2)\Phi(x), \quad \Phi(-x) = -\Phi(x).$$

It is easy to show that $\Phi(x) < 0$, for all $x > 0$, and $\Phi(0) = 0$. Thus the unique singular points of the differential system are: $(0, 0), (a/2, 0), (a, 0)$.

The quantities

$$\sigma\,(x_0, 0) = \operatorname{div}(P, Q)\,(x_0, 0)\,, \quad \Delta\,(x_0, 0) = \begin{vmatrix} P_x & Q_x \\ P_y & Q_y \end{vmatrix} (x_0, 0)\,,$$

for system (39) are such that:

$$\begin{aligned}
\sigma(0,0) = &\quad \sigma(a,0), \quad \Delta(0,0) = \Delta(a,0),\\
\sigma(0,0) = &\quad (-1)^S a \prod_{j=1}^{S} r_j^2 \neq 0,\\
\Delta(0,0) = &\quad a^2 \prod_{j=1}^{S}(a^2 - r_j^2)\left(\prod_{j=1}^{S}(a^2 - r_j^2) + (-1)^S \prod_{j=1}^{S} r_j^2\right) > 0,\\
\sigma^2(0,0) - 4\Delta(0,0) = &\quad a^2\left(2\prod_{j=1}^{S}(a^2 - r_j^2) - (-1)^S \prod_{j=1}^{S} r_j^2\right)^2\\
&\quad -8a^2\left(\prod_{j=1}^{S}(a^2 - r_j^2)\right)^2 < 0,
\end{aligned}$$

and

$$\begin{aligned}
\sigma(\frac{a}{2},0) = &\quad a\prod_{j=1}^{S}(\frac{a^2}{4} - r_j^2),\\
\Delta(\frac{a}{2},0) = &\quad -\frac{a^4}{4}\prod_{j=1}^{S}(\frac{a^2}{4} - r_j^2)\sum_{k=1}^{S}\prod_{j\neq k}^{S}\Big((\frac{a^2}{4} - r_j^2\Big).
\end{aligned}$$

It is well known that if $\Delta(x_0, y_0) < 0$ then the singular point (x_0, y_0) is a saddle, and if $\Delta(x_0, y_0) > 0$ and $\sigma^2(x_0, y_0) - 4\Delta(x_0, y_0) < 0$ then the singular point is a foci which is stable if $\sigma(x_0, y_0) < 0$ and unstable if $\sigma(x_0, y_0) > 0$. If $a > 2r_S$, then $\Delta(a/2, 0) < 0$, and, as a consequence the singular point $(a/2, 0)$ is a saddle. The other two singular points are focii their stability depend on the parity of S (for more details see [1]). Thus the given circles are isolated periodic solutions of the differential system (39), i.e. are limit cycles [11]. □

Proposition 22. *The differential system*

$$\begin{aligned}
\dot{x} = &\quad y\,\nu_\varepsilon,\\
\dot{y} = &\quad -x\,\nu_\varepsilon + a\prod_{n=1}^{S+1}(x^2 + y^2 - \frac{1}{n^2}),
\end{aligned} \tag{40}$$

where ε is small parameter, $a > 2$ and

$$\nu_\varepsilon = \prod_{n=1}^{S+1}(x^2 + y^2 - \frac{1}{n^2}) - ((x-a)^2 + y^2 + \varepsilon y + (S+1)^{-2})\prod_{n=1}^{S}((x-a)^2 + y^2 - \frac{1}{n^2}),$$

of degree $m = 2S + 2$ has only two equilibrium points which are foci, Moreover the circles $x^2 + y^2 - \dfrac{1}{n^2} = 0$ for $n = 1, \ldots, S+1$, and $(x-a)^2 + y^2 - \dfrac{1}{n^2} = 0$ for $n = 1, \ldots, S$ are limit cycles of the system.

Proof. We claim that system (40) has two equilibrium points which are foci. Now we prove the claim. First we show that there are no singular points (x_0, y_0) on the curve $\nu_\varepsilon(x, y) = 0$, i.e. $\nu_\varepsilon(x_0, y_0) \neq 0$ for all (x_0, y_0). Otherwise from (40) we

should have $\prod_{n=1}^{S+1}(x_0^2+y_0^2-\frac{1}{n^2})=0$, and consequently $((x_0-a)^2+y_0^2+\varepsilon y_0+1/(S+1)^2)\prod_{n=1}^{S}((x_0-a)^2+y_0^2-\frac{1}{n^2})=0$, and this a contradiction because these last two curves do not intersect. Thus the singular points of system (40) are of the type $(x_0,0)$ where x_0 is a root of the polynomial

$$L(x)=-(x-a)\prod_{n=1}^{S+1}(x^2-\frac{1}{n^2})+x((x-a)^2+(S+1)^{-2})\prod_{n=1}^{S}((x-a)^2-\frac{1}{n^2})$$

of degree $2S+1$. By considering that $a>2$ then we obtain that

$$\begin{aligned}
L(0) &= (-1)^{1+S}a\prod_{n=1}^{S+1}\frac{1}{n^2}, \quad L(a)=(-1)^{S}a\prod_{n=1}^{S+1}\frac{1}{n^2},\\
L(a/2) &= \frac{a^3}{4}\prod_{n=1}^{S}(\frac{a^2}{4}-\frac{1}{n^2})>0,\\
L(-1/S) &= -1/S((a-1/S)^2+1/(S+1)^2)\prod_{n=1}^{S}((1/S+a)^2-\frac{1}{n^2})<0,\\
L(1/S) &= 1/S((a-1/S)^2+1/(S+1)^2)\prod_{n=1}^{S}((1/S-a)^2-\frac{1}{n^2})>0,\\
L(a-1/S) &= 1/S\prod_{n=1}^{S+1}((1/S-a)^2-\frac{1}{n^2})>0,\\
L(a+1/S) &= -1/S\prod_{n=1}^{S+1}((1/S+a)^2-\frac{1}{n^2})<0,
\end{aligned}$$

and we obtain that

$$\begin{aligned}
L(x)<0 \qquad &\text{for} \quad x<-1/S,\\
L(x)<0 \qquad &\text{for } x>a+1/S, \quad ,\\
L(-\frac{1}{S})L(\frac{1}{S}) < \quad 0, \quad &L(a-\frac{1}{S})L(a+\frac{1}{S})<0,
\end{aligned}$$

and by considering that the minimum of $L(x)$ is reached in a point which is close of the point $x=a/2$ and its value is positive, then we have proved that in the intervals $(-1/S,1/S)$ and $(a-1/S,a+1/S)$ there exist only one real root. These roots approximately have the coordinates

$$\left((-1)^S\,10^{2-2S},0\right), \quad \left(a+(-1)^{S+1}\,10^{2-2S},0\right),$$

and clearly tends to the point $(0,0)$ and $(a,0)$ when S tends to infinity.

These singular points are foci. Indeed, by considering that if $(x_0,0)$ is a singular point of (40) then the linear part of this differential system are

$$\begin{aligned}\dot{x} = &\quad \nu_0(x_0,0)\,y,\\ \dot{y} = &\quad (-\nu_0(x_0,0)+2x_0(a-x_0)\sigma(x_0))\,x+\varepsilon\,x_0\prod_{n=1}^{S}((x_0-a)^2-\frac{1}{n^2})\,y,\end{aligned}$$

where

$$\sigma(x)=\sum_{j=1}^{S+1}\prod_{\substack{m=1\\ m\neq j}}^{S+1}\left(x^2-\frac{1}{m^2}\right)+\left((x-a)^2+1/(S+1)^2\right)\sum_{j=1}^{S}\prod_{\substack{m=1\\ m\neq j}}^{S}\left((x-a)^2-\frac{1}{m^2}\right)$$

consequently the point $(x_0,0)$ is a foci because the roots λ_1 and λ_2 such that

$$\begin{vmatrix}\lambda & -\nu_0(x_0,0)\\ \nu_0(x_0,0)-2x_0(a-x_0)\sigma(x_0) & \lambda-\varepsilon\,x_0\prod_{n=1}^{S}((x_0-a)^2-\frac{1}{n^2})\end{vmatrix}=0$$

are complex numbers. The stability of this singular point depend of the sign of the number $\varepsilon\,x_0\prod_{n=1}^{S}((x_0-a)^2-\frac{1}{n^2})$. In short the claim is proved.

For $\varepsilon=0$ we obtain that the singular points are centers. Indeed from (40) follows that

$$\begin{aligned}\frac{d}{dt}(x^2+y^2) = &\quad 2ay\prod_{n=1}^{S+1}(x^2+y^2-\frac{1}{n^2}),\\ \frac{d}{dt}((x-a)^2+y^2) = &\quad 2ay((x-a)^2+y^2+\varepsilon y+(S+1)^{-2})\cdot\\ &\quad \prod_{n=1}^{S}((x-a)^2+y^2-\frac{1}{n^2}),\end{aligned}$$

when $\varepsilon=0$ follows that

$$\frac{\dfrac{d}{dt}(x^2+y^2)}{\displaystyle\prod_{n=1}^{S+1}(x^2+y^2-\frac{1}{n^2})}-\frac{\dfrac{d}{dt}((x-a)^2+y^2)}{((x-a)^2+y^2+(S+1)^{-2})\displaystyle\prod_{n=1}^{S}((x-a)^2+y^2-\frac{1}{n^2})}=0.$$

Hence we deduce the existence of the first integral

$$H(x,y)=\frac{\displaystyle\prod_{n=1}^{S}\left|(x-a)^2+y^2-\frac{1}{n^2}\right|^{\mu_n}}{\displaystyle\prod_{n=1}^{S+1}\left|x^2+y^2-\frac{1}{n^2}\right|^{\lambda_n}}e^{\arctan((x-a)^2+y^2+(S+1)^{-2})}$$

where $\mu_n=\displaystyle\prod_{\substack{l=1\\ l\neq n}}^{S}\left(\frac{1}{n^2}-\frac{1}{l^2}\right)$ and $\lambda_n=\displaystyle\prod_{\substack{l=1\\ l\neq n}}^{S+1}\left(\frac{1}{n^2}-\frac{1}{l^2}\right)$. Since the integral or its inverse is defined on the circles $(x-a)^2+y^2-\frac{1}{n^2}=0$ for $n=1,\ldots,S$ and $x^2+y^2-\frac{1}{n^2}=0$, for $n=1,\ldots,S+1$. These periodic solutions cannot be limit

cycles. Finally by the analyticity of the Poincaré map it follows that the two singular points are centers. □

Proof of Theorem 4. From Theorem 3 follows that the polynomial vector field of degree S with S invariant circles is Darboux integrable without limit cycles. From proposition 21 and 22 follows that there exist polynomial planar vector fields of degree S with $S-1$ invariant circles which are limit cycles. □

Remark 23. *There exist other configuration of polynomial vector field of degree S with $S-1$ invariant circles as limit cycles different from the circles given in Proposition 21 and 22. For example for $S=4$ we have the following quartic vector field with 3 invariant circles as limit cycles.*

Example 24. *The polynomial vector fields of degree four*

$$\begin{aligned}\dot{x} = \;& 8x^4-(28y+48)x^3+(274y+56)x^2+(-28y^3+144y^2\\ & -1028y+48)x-8y^4+82y^3-216y^2+1070y-64,\\ \dot{y} = \;& 14x^4+(16y-178)x^3-(144y-792)x^2\\ & +(16y^3+14y^2+272y-1166)x-14y^4\\ & +48y^3-236y^2-48y+250,\end{aligned}$$

admits the following three invariant circles which are limit cycles

$$x^2+y^2-1=0,\quad (x-4)^2+(y-4)^2-1=0,\quad (x-3)^2+y^2-1=0,$$

with cofactors

$$\begin{aligned}&2(-24x^2+48xy+24y^2+32x-125y+4x^3-7yx^2+4xy^2-7y^3),\\ &2(-36x^2+48xy+12y^2+68x-65y-12+4x^3-7yx^2+4xy^2-7y^3),\\ &2(-12x^2+48xy+36y^2-4x-185y+12+4x^3-7yx^2+4xy^2-7y^3),\end{aligned}$$

6. Other results related with invariant circles of the polynomial planar vector field

Theorem 25. *The following statement hold.*

(a) *Every configuration of $S\geq 2$ circles in the plane is realizable for a planar polynomial vector field of degree $m=2S$ if m is even, and $m=2S-1$ if m is odd.*

(b) *Every configuration of $S\geq 2$ pairwise disjoint circles in the plane is realizable by algebraic limit cycles for a planar polynomial vector field of degree $m=2S$. These limit cycles are the unique limit cycles of the polynomial vector field.*

Proof. Let $g_j=(x-a_j)^2+(y-b_j)^2-r_j^2=0$, for $j=1,\dots,S$, be the given circles We consider a polynomial vector fields (37) of the degree $2S$, with $\lambda_{S+1}=a$, $\lambda_{S+2}=b$, where a and b are arbitrary constants.

On the other hand we consider a polynomial vector fields (37) of the degree $2S-1$, taking

$$\begin{aligned}\lambda_{S+1} = \;& \lambda_{S+2}=0,\quad \lambda_1=C_1-\sum_{j=2}^{S}(A_jx+B_jy),\\ \lambda_j = \;& A_jx+B_jy+C_j,\quad \text{for}\quad j=2,\dots,S,\end{aligned}$$

where C_1, A_j, B_j and C_j for $j = 2, \ldots, S$, are arbitrary constants. Under these conditions system (37) is a polynomial vector field of degree $2S-1$. Thus statement (a) of Theorem 25 is proved.

We study the case when the vector field (37) has the degree $m = 2S$ with

$$\lambda_1 = \ldots = \lambda_S = \lambda = (x + y + C)/2, \qquad \lambda_{S+1} = -1, \quad \lambda_{S+2} = 1,$$

where C is a constant such that the straight line $x + y + C = 0$ does not intersect the given circles $g_j = 0$. Denote by $g = \prod_{j=1}^{S} g_j$, then (37) takes the form

$$\dot{x} = g - (x + y + C)g_y, \qquad \dot{y} = g + (x + y + C)g_x,$$

In view of Theorem 19 we obtain the proof of statement (b) of Theorem 25. □

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