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Integrability of the constrained rigid body

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Abstract The integrability theory for the differential equations, which describe the motion of an unconstrained rigid body around a fixed point is well known. When there are constraints the theory of integrability is incomplete. The main objective of this paper is to analyze the integrability of the equations of motion of a constrained rigid body around a fixed point in a force field with potential $U(\gamma) = U(\gamma_1, \gamma_2, \gamma_3)$. This motion subject to the constraint $\langle \nu, \omega \rangle = 0$ with ν is a constant vector is known as the Suslov problem, and when $\nu = \gamma$ is the known Veselova problem, here $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity and $\langle\, ,\, \rangle$ is the inner product of $\mathbb{R}^3$.

We provide the following new integrable cases.

(i) The Suslov's problem is integrable under the assumption that ν is an eigenvector of the inertial tensor I and the potential is such that

$$U = -\frac{1}{2I_1 I_2}\left(I_1 \mu_1^2 + I_2 \mu_2^2\right),$$

where I_1, I_2, and I_3 are the principal moments of inertia of the body, μ_1 and μ_2 are solutions of the first-order partial differential equation

$$\gamma_3\left(\frac{\partial \mu_1}{\partial \gamma_2} - \frac{\partial \mu_2}{\partial \gamma_1}\right) - \gamma_2 \frac{\partial \mu_1}{\partial \gamma_3} + \gamma_1 \frac{\partial \mu_2}{\partial \gamma_3} = 0.$$

(ii) The Veselova problem is integrable for the potential

$$U = -\frac{\Psi_1^2 + \Psi_2^2}{2(I_1 \gamma_2^2 + I_2 \gamma_1^2)},$$

where Ψ_1 and Ψ_2 are the solutions of the first-order partial differential equation

$$(I_2 - I_1)\gamma_1\gamma_2\left\langle \gamma, \frac{\partial \Psi_2}{\partial \gamma}\right\rangle + I_1\gamma_2 \frac{\partial \Psi_2}{\partial \gamma_1} - I_2 \gamma_1 \frac{\partial \Psi_2}{\partial \gamma_2} - p\left(\gamma_3\left\langle \gamma, \frac{\partial \Psi_1}{\partial \gamma}\right\rangle - \frac{\partial \Psi_1}{\partial \gamma_3}\right) = 0,$$

where $p = \sqrt{I_1 I_2 I_3 \left(\frac{\gamma_1^2}{I_1} + \frac{\gamma_2^2}{I_2} + \frac{\gamma_3^2}{I_3}\right)}$.

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