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A new approach to the vakonomic mechanics

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Abstract The aim of this paper was to show that the Lagrange–d'Alembert and its equivalent the Gauss and Appel principle are not the only way to deduce the equations of motion of the nonholonomic systems. Instead of them we consider the generalization of the Hamiltonian principle for nonholonomic systems with non-zero transpositional relations. We apply this variational principle, which takes into the account transpositional relations different from the classical ones, and we deduce the equations of motion for the nonholonomic systems with constraints that in general are nonlinear in the velocity. These equations of motion coincide, except perhaps in a zero Lebesgue measure set, with the classical differential equations deduced with the d'Alembert–Lagrange principle. We provide a new point of view on the transpositional relations for the constrained mechanical systems: the virtual variations can produce zero or non-zero transpositional relations. In particular, the independent virtual variations can produce non-zero transpositional relations. For the unconstrained mechanical systems, the virtual variations always produce zero transpositional relations. We conjecture that the existence of the nonlinear constraints in the velocity must be sought outside of the Newtonian mechanics. We illustrate our results with examples.

Keywords Variational principle · Generalized Hamiltonian principle · d'Alembert–Lagrange principle · Constrained Lagrangian system · Transpositional relations · Vakonomic mechanic · Equation of motion · Vorones system · Chapligyn system · Newtonian model

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1 Introduction

The history of nonholonomic mechanical systems is long and complex and goes back to the 19 century, with important contribution by Hertz [17] (1894), Ferrers [11] (1871), Vierkandt [51] (1892) and Chaplygin [7] (1897).

The nonholonomic mechanic is a remarkable generalization of the classical Lagrangian and Hamiltonian mechanic. The birth of the theory of dynamics of nonholonomic systems occurred when Lagrangian-Euler formalism was found to be inapplicable for studying