Applied Mathematics and Computation 219 (2013) 8358–8365

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A note on the periodic orbits of a kind of Duffing equations

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ARTICLE INFO

Keywords:
Periodic orbit
Duffing differential equation
Averaging theory

ABSTRACT

We study the periodic orbits of the modified Duffing differential equation $\ddot{y} + ay - \varepsilon y^3 = \varepsilon h(y, \dot{y})$ with $a > 0, \varepsilon$ a small parameter and h a $\mathcal{C}^2$ function in its variables.

1. Introduction

In a paper published in 1922, Hamel [8] provided the first general results for the existence of periodic solutions of the periodically forced pendulum equation

$$\ddot{y} + a \sin y = b \sin t. \tag{1}$$

This equation was the main subject of a monograph published four years earlier by Duffing [5], who had restricted his study to the approximate determination of the periodic solutions for the following approximation of Eq. (1):

$$\ddot{y} + ay - cy^3 = b \sin t,$$

which now is known as the *Duffing differential equation*. For more details on the history of these differential equations see the paper of Mawhin [10]. Many of the 190 references quoted in this last paper are on the periodic orbits of different kind of Duffing equations, and from its publication many new papers working on these type of periodic orbits also have been published, see for instance the papers [3,4,13] and the quoted references in there. Similar non-autonomous differential equations have been studied in [7].

Here we consider the following *modified Duffing differential equation*

$$\ddot{y} + ay - \varepsilon y^3 = \varepsilon h(y, \dot{y}), \tag{2}$$

with $a > 0, \varepsilon$ a small parameter and h a $\mathcal{C}^2$ function in its variables.

We recall that $\bar{k}$ is a *simple zero* of a real function $f(k)$ if $f(\bar{k}) = 0$ and $(df/dk)(\bar{k}) \neq 0$.

Our main result is the following.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small and for every simple positive zero $\bar{k}$ of the function*

$$f(k) = \int_0^{2\pi} h\left(k \cos \theta, \frac{k \sin \theta}{\sqrt{a}}\right) \cos \theta d\theta,$$

the modified Duffing differential equation (2) has a periodic orbit $y(t, \varepsilon)$ such that when $\varepsilon \to 0$ we have that $(\dot{y}(t, \varepsilon),\ y(t, \varepsilon))$ tends to the periodic orbit given by the ellipse

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0096-3003/$ - see front matter © 2012 Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.amc.2012.11.087