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Topological entropy and periods of graph maps

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We give a complete description of the set of periods of continuous self-maps on graphs which have zero topological entropy and have all their branching points fixed. As a corollary, we get a criterion for positive topological entropy for continuous self-maps on graphs. We also prove that the results are optimal.

Keywords: graph maps; periodic points; periods; topological entropy

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1. Introduction

If a function f is a self-map on some space M, then one of the simplest dynamical invariants one would like to describe is the set of periodic points. A point $x \in M$ is *periodic* if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$, and the smallest n with this property is called the *(least) period of x*. The *set of periods of f* is

$$\mathrm{Per}(f) = \{n \in \mathbb{N} : n \text{ is the period of some periodic point } x \text{ of } f\}.$$

This set is well understood in the case of continuous maps on some simple spaces like the interval ([11]), circle ([2] for example), '*Y*' ([1]), n-od ([4]), trees with branching points fixed ([5]) or trees in general ([3]). However, if the space becomes more complicated, or higher dimensional, it is very difficult to describe the set of periods for a general map, so one usually has to add some restrictions.

If M is a metric space and f is continuous, then one can define the *topological entropy* of f, or simply the *entropy* of f, denoted by $h(f)$, as an invariant measuring the complexity of the orbits of f (see [9] for example). The presence of positive entropy implies chaotic behaviour, so one would like to know when this happens.

A *graph* is a union of finitely many *vertices* – points – and *edges*, which are homeomorphic to the closed interval, and have mutually disjoint interiors. The endpoints of the edges are vertices (not necessarily different). An *endpoint of a graph* is a vertex which is the endpoint of only one edge. A *branching point* is a vertex which is the endpoint of at least three edges (if an edge has both endpoints at that vertex, we count the edge twice). An *ending edge* is an edge which contains an endpoint of the graph. A *path* in graph G is the image of a continuous map from [0,1] to G. If the endpoints coincide, and the path is not homotopic to a point, then it is called a *cycle*. A *tree* is a graph which does not contain any cycle. One can put a metric on any graph, first on the edges from the

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