ON THE LIMIT CYCLES SURROUNDING A DIAGONALIZABLE LINEAR NODE WITH HOMOGENEOUS NONLINEARITIES

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Abstract. In this paper we study the existence and non–existence of limit cycles for the class of polynomial differential systems of the form

$$\dot{x} = \lambda x + P_n(x,y), \qquad \dot{y} = \mu y + Q_n(x,y),$$

where P_n and Q_n are homogeneous polynomials of degree n.

1. Introduction and statement of the main results

A *polynomial differential system* in $\mathbb{R}^2$ is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials in the variables x and y with real coefficients. Then $m = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system.

As usual a *limit cycle* of a system (1) is an isolated periodic solution in the set of all periodic solutions of system (1). Limit cycles of planar differential systems were defined by Poincaré [21] and started to be studied intensively at the end of the 1920s by van der Pol [22], Liénard [12] and Andronov [1].

In the qualitative theory of the polynomial differential equations in the plane $\mathbb{R}^2$ one of the more difficult problems is the study of their limit cycles. Thus the second part of the unsolved 16–th Hilbert problem [13] asked for an upper bound on the maximum number of limit cycles for the polynomial differential systems of a given degree in function of this degree, see for more details the surveys [14] and [11].

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In this paper for the class of polynomial differential systems in $\mathbb{R}^2$ of the form

$$\dot{x} = P_1(x,y) + P_n(x,y), \qquad \dot{y} = Q_1(x,y) + Q_n(x,y), \tag{2}$$

where $n > 1$, and $P_k(x,y)$ and $Q_k(x,y)$ are homogeneous polynomials of degree k, we want to study the existence and non–existence of limit cycles.

For the polynomial differential systems (2) having a linear focus at the origin of coordinates of the form

$$\dot{x} = \lambda x - y + P_n(x,y), \qquad \dot{y} = x + \lambda y + Q_n(x,y),$$

their limit cycles have been studied intensively, see for instance [3, 4, 5, 6, 8, 9, 10, 15, 17, 18, 20]. But there are very few results on the limit cycles of the polynomial differential systems having a linear node at the origin of coordinates of the form

$$\dot{x} = \lambda x + P_n(x,y), \qquad \dot{y} = \mu y + Q_n(x,y), \tag{3}$$

with $\lambda\mu > 0$.

Recently in [2] the polynomial differential systems (3) with $\lambda = \mu$ and $n > 1$ have been analyzed, proving that if n is odd such systems have at most one limit cycle, and if n is even then they have no limit cycles. On the other hand, in Proposition 6.3 and Remark 6.4 of the paper [7] are examples of systems (3) having two, one or zero limit cycles surrounding the origin. Finally, when $\lambda \neq \mu$ and $\lambda\mu > 0$ in [16] the authors provide sufficient conditions for the non–existence of limit cycles, or for the existence of one or two limit cycles.

Using polar coordinate $x = r\cos(\theta)$ and $y = \sin\theta$ system (3) becomes

$$\dot{r} = f_0(\theta)r + f(\theta)r^n, \qquad \dot{\theta} = g_0(\theta) + g(\theta)r^{n-1}, \tag{4}$$

and in the region $R = \{(r,\theta) : g_0(\theta) + g(\theta)r^{n-1} > 0\}$ it can be studied using the differential equation

$$\frac{dr}{d\theta} = \frac{f_0(\theta)r + f(\theta)r^n}{g_0(\theta) + g(\theta)r^{n-1}}, \tag{5}$$

where

$$\begin{aligned}
f_0(\theta) &= \lambda\cos^2\theta + \mu\sin^2\theta,\\
g_0(\theta) &= (\mu-\lambda)\cos\theta\sin\theta,\\
f(\theta) &= \cos\theta P_n(\cos\theta,\sin\theta) + \sin\theta Q_n(\cos\theta,\sin\theta),\\
g(\theta) &= \cos\theta Q_n(\cos\theta,\sin\theta) - \sin\theta P_n(\cos\theta,\sin\theta),\\
P_n(x,y) &= \sum_{i=0}^{i=n} a_{(n-i)i}x^{n-i}y^i,\\
Q_n(x,y) &= \sum_{i=0}^{i=n} b_{(n-i)i}x^{n-i}y^i.
\end{aligned}$$

Theorem 1. *The polynomial differential system* (3) *with* $n \geq 2$ *has no limit cycles surrounding the origin in the region* R *if one of the following conditions holds.*

(a) $f = 0$.
(b) $g = 0$.
(c) $a_{0n} = 0$.
(d) $b_{n0} = 0$.
(e) $g = g_0 f/f_0$.
(f) $(nfg + (2-n)f_0 g)^2 - 4fgf_0g_0 \leq 0$.
(g) $((2n-1)fg_0 - (2n-3)f_0 g)^2 - 4f_0g_0fg \leq 0$.

The polynomial differential system (3) *has at most one limit cycle surrounding the origin if the following condition holds*

(h) $((2n-1)f_0 g - (2n-3)fg_0)^2 - 4f_0g_0fg \leq 0$.

Theorem 1 is proved in the next section.

2. Proofs

For proving Theorem 1 we need the following two lemmas due to Lloyd [19].

Lemma 2. *We have in a simply connected open set* V *containing the origin the differential system in polar coordinates*

$$\dot r = S_1(r,\theta), \quad \dot\theta = S_2(r,\theta), \tag{6}$$

where S_1 *and* S_2 *are* C^1 2π*-periodic functions such that* $S_1(0,\theta) = 0$ *for all* θ*, and* $S_2(r,\theta) > 0$ *in* V*. The differential system* (6) *is equivalent* *to the differential equation*

$$\frac{dr}{d\theta} = \frac{S_1(r,\theta)}{S_2(r,\theta)} = S(r,\theta). \tag{7}$$

Therefore, if

$$\frac{\partial S}{\partial r} \equiv 0, \quad \frac{\partial S}{\partial r} \leqslant 0, \text{ or } \frac{\partial S}{\partial r} \geqslant 0 \tag{8}$$

in V, then the differential system (6) *has no limit cycles in V.*

Lemma 3. *Consider the differential system* (6) *defined in an annular region A that encircles the origin and where $S_2(r,\theta) > 0$. Then in A, the differential system* (6) *is equivalent to the differential equation* (7)*. If* (8) *hold in A, then the differential system 6 has at most one limit cycle in A.*

Proof statement (*a*) *of Theorem 1.* If $f = 0$ equation (5) becomes

$$\frac{dr}{d\theta} = \frac{f_0(\theta)r}{g_0(\theta) + g(\theta)r^{n-1}}.$$

Since $\lambda >$ and $\mu > 0$ this last equation does not change sign in the region C. The solution $r(\theta)$ of this equation increases or decreases, so these solutions cannot be periodic in the region R, and consequently the polynomial differential system (3) has no limit cycles In R. □

Proof statement (*b*) *of Theorem 1.* Since $g = 0$ the differential equation (4) becomes

$$\dot{r} = f_0(\theta)r + f(\theta)r^n, \qquad \dot{\theta} = g_0(\theta).$$

The straight lines $\theta = 0$ and $\theta = \pi/2$ are invariant for system (3). So this system cannot have limit cycles surrounding the origin. This completes the proof of this statement. □

Proof statement (*c*) *of Theorem 1.* Since $a_{0n} = 0$ the differential system (3) has the straight line $x = 0$ invariant, consequently this system has no limit cycles surrounding the origin. □

The same argument used in the proof of statement (c) proves statement (d).

Proof statement (e) of Theorem 1. Since $g = fg_0/f_0$ system (4) becomes

$$\dot{r} = f_0(\theta)r + f(\theta)r^n, \qquad \dot{\theta} = g_0(\theta)(1 + \frac{f(\theta)}{f_0(\theta)})r^{n-1}.$$

So the proof ends following the same argument used in the proof of statement (b). ☐

Proof statement (f) of Theorem 1. Let

$$S(r,\theta) = \frac{f_0 r + f r^n}{g_0 + g r^{n-1}},$$

defined in the simply connected region R. The derivative of S with respect to r is

$$\frac{\partial S}{\partial r} = \frac{f_0 g_0 + (nfg_0 + (2-n)f_0 g)r^{n-1} + fgr^{2n-2}}{(g_0 + gr^{n-1})^2}.$$

Since $(nfg+(2-n)f_0g)^2 - 4fgf_0g_0 \leq 0$ the numerator of $\partial S/\partial r$ does not change of sign, and we can apply Lemma 2 to the differential equation (5), and the proof of this statement follows. ☐

Proof statement (g) of Theorem 1. Doing the change of variables $R = \sqrt{r}$ in the region C, the differential equation (5) becomes

$$\frac{dR}{d\theta} = \frac{f_0 R + f R^{2n-1}}{2(g_0 + g R^{2n-2})} = S(R,\theta). \tag{9}$$

The derivative of S with respect to R is

$$\frac{\partial S}{\partial R} = \frac{f_0 g_0 + ((2n-1)fg_0 - (2n-3)f_0 g)R^{2n-2} + fgR^{4n-4}}{2(g_0 + gR^{2n-2})^2}.$$

Since $((2n-1)fg_0 - (2n-3)f_0g)^2 - 4f_0g_0fg \leq 0$ the numerator of $\partial S/\partial R$ does not change of sign,, and again we can apply Lemma 2 to the differential equation (9), and statement (g) is proved. ☐

Proof statement (h) of Theorem 1. Doing the change of variables $R = 1/\sqrt{r}$ in the region R the differential equation (5) becomes

$$\frac{dR}{d\theta} = \frac{R(f_0 R^{2n-2} + f)}{2(g_0 R^{2n-2} + g)} = S(R,\theta). \tag{10}$$

So the derivative of S with respect to R is

$$\frac{\partial S}{\partial R} = -\frac{fg + ((2n-1)f_0 g - (2n-3)fg_0)R^{2n-2} + f_0 g_0 R^{4n-4}}{2(g_0 R^{2n-2} + g)^2}.$$

The image of the region R under the map $r \to 1/\sqrt{r}$ is an annular region A, one of the boundaries of this annulus is the infinity. Since $((2n-1)f_0g - (2n-3)fg_0)^2 - 4f_0g_0fg \leq 0$ the numerator of $\partial S/\partial R$ does not change of sign, we can apply Lemma 3 in the annular region A to the differential equation (10), and this completes the proof of this statement. ☐

In short Theorem 1 is proved.

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