ON THE LIMIT CYCLES OF THE PIECEWISE DIFFERENTIAL SYSTEMS FORMED BY A LINEAR FOCUS OR CENTER AND A QUADRATIC WEAK FOCUS OR CENTER

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ABSTRACT. While the limit cycles of the discontinuous piecewise differential systems formed by two linear differential systems separated by one straight line have been studied intensively, and up to now there are examples of these systems with at most 3 limit cycles. There are almost no works studying the limit cycles of the discontinuous piecewise differential systems formed by one linear differential system and a quadratic polynomial differential system separated by one straight line.

In this paper using the averaging theory up to seven order we prove that the discontinuous piecewise differential systems formed by a linear focus or center and a quadratic weak focus or center separated by one straight line can have 8 limit cycles. More precisely, at every order of the averaging theory from order one to order seven we provide the maximum number of limit cycles that can be obtained using the averaging theory.

1. Introduction and statement of the main results

Piecewise differential systems are provided as one of the most remarkable non-smooth dynamical systems and widely applied in various scientific domains of studies such as engineering, electronics, and physics [1, 2, 8, 12, 15, 16, 24, 26, 27]. Since the 1930s many books and papers study the piecewise differential systems, mainly due to their applications to mechanics and electrical circuits, see for instance [6, 7, 25, 28]. The more studied piecewise differential systems are the continuous and discontinuous piecewise differential systems separated by a straight-line, see for instance [4, 9, 10, 11, 17, 18, 19, 20, 21, 22, 23].

A limit cycle is an isolated periodic orbit in the set of all periodic orbits of a differential system. Limit cycles play a main role in the qualitative theory of the differential systems, and also in the discontinous piecewise differential systems. The singular point $p \in \mathbb{R}^2$ is a center of a planar differential system if there is a neighborhood U of p where all the orbits of $U \setminus \{p\}$ are periodic.

Our objective is to study the limit cycles which bifurcate from the periodic orbits of the linear differential center $\dot{x} = -y,\ \dot{y} = x$, when we perturb this center by discontinuous piecewise differential systems separated by the straight line $y = 0$ and formed by linear differential focus or center

$$\dot{x} = \alpha x + \beta y + \gamma,\ \dot{y} = -\beta x + \alpha y + \delta \tag{1}$$

defined in $y \geq 0$, and quadratic weak focus or center at the origin

$$\dot{x} = -y - bx^2 - cxy - dy^2,\ \dot{y} = x + ax^2 + Axy - ay^2, \tag{2}$$

defined in $y \leq 0$. For more details on the quadratic weak focus or center see Lemma 8.14 of [5].

Our main result is the following theorem.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small the maximum number of limit cycles of the piecewise differential systems obtained perturbing the linear differential center $\dot{x} = -y$, $\dot{y} = x$ by the discontinuous*

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piecewise differential system formed by systems (1) *and* (2) *obtained using averaging theory up to seven order is eight.*

Theorem 1 is proved in section 3. We note that in general to study analytically the limit cycles is a very difficult task, here we do this study using the new theory of averaging for discontinuos piecewise differential systems developed in [14], a summary of this theory is given in section 2.

2. The averaging theory up to order 7 for computing limit cycles

In this section we present the basic results from the averaging theory for computing the periodic solutions of discontinuous piecewise differential systems that we shall need for proving the main results of this paper. This improvement of the classical averaging theory for computing limit cycles of planar discontinuous piecewise differential systems was developed in [14], a summary of this theory is given in below. We consider discontinous differential systems of the form

$$\dot{r}(\theta) = \begin{cases} F^+(\theta, r, \varepsilon) & \text{if } 0 \le \theta \le \pi, \\ F^-(\theta, r, \varepsilon) & \text{if } \pi \le \theta \le 2\pi, \end{cases} \tag{3}$$

where $F^{\pm}(\theta, r, \varepsilon) = \sum_{i=0}^{7} \varepsilon^i F_i^{\pm}(\theta, r) + \varepsilon^8 R^{\pm}(\theta, r, \varepsilon)$, with $\theta \in \mathbb{S}^1$ and $r \in D$, where D is an open interval of $\mathbb{R}^+$, and ε is a small real parameter.

From [14] we define the following functions $y_i^{\pm}(t, r)$ for $k = 1, 2, 3, 4, 5, 6, 7$ related to system (3) :

$$\begin{aligned}
y_1^{\pm}(s, r) &= \int_0^s F_1^{\pm}(t, r)dt, \\
y_2^{\pm}(s, r) &= \int_0^s [2F_2^{\pm}(t, r) + 2\partial F_1^{\pm}(t, r) y_1^{\pm}(t, r)]dt, \\
y_3^{\pm}(s, r) &= \int_0^s [6F_3^{\pm}(t, r) + 6\partial F_2^{\pm}(t, r) y_1^{\pm}(t, r) \\
&\qquad + 3\partial^2 F_1^{\pm}(t, r) y_1^{\pm}(t, r)^2 + 3\partial F_1^{\pm}(t, r) y_2^{\pm}(t, r)]dt, \\
y_4^{\pm}(s, r) &= \int_0^s [24F_4^{\pm}(t, r) + 24\partial F_3^{\pm}(t, r) y_1^{\pm}(t, r) \\
&\qquad + 12\partial^2 F_2^{\pm}(t, r) y_1^{\pm}(t, r)^2 + 12\partial F_2^{\pm}(t, r) y_2^{\pm}(t, r) \\
&\qquad + 12\partial^2 F_1^{\pm}(t, r) y_1^{\pm}(t, r) y_2^{\pm}(t, r) \\
&\qquad + 4\partial^3 F_1(t, r) y_1^{\pm}(t, r)^3 + 4\partial F_1(t, r) y_3^{\pm}(t, r)]dt,
\end{aligned}$$

$$
\begin{aligned}
y_5^{\pm}(s,r) = \int_0^s & [120F_5^{\pm}(t,r) + 120\partial F_4^{\pm}(t,r)y_1^{\pm}(t,r) \\
& + 60\partial^2 F_3^{\pm}(t,r)y_1^{\pm}(t,r)^2 + 60\partial F_3^{\pm}(t,r)y_2^{\pm}(t,r) \\
& + 60\partial^2 F_2^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r) + 20\partial^3 F_2^{\pm}(t,r)y_1^{\pm}(t,r)^3 \\
& + 20\partial F_2^{\pm}(t,r)y_3^{\pm}(t,r) + 20\partial^2 F_1^{\pm}(s,r)y_1^{\pm}(t,r)y_3^{\pm}(t,r) \\
& + 15\partial^2 F_1^{\pm}(t,r)y_2^{\pm}(t,r)^2 + 30\partial^3 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2 y_2^{\pm}(t,r)) \\
& + 5\partial^4 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^4 + 5\partial F_1^{\pm}(t,r)y_4^{\pm}(t,r)]dt, \\
y_6^{\pm}(s,r) = 720\int_0^s & [F_6^{\pm}(t,r) + \partial F_5^{\pm}(t,r)y_1^{\pm}(t,r) + \frac{1}{2}\partial^2 F_4^{\pm}(t,r)y_1^{\pm}(t,r)^2 \\
& + \frac{1}{2}\partial F_4^{\pm}(t,r)y_2^{\pm}(t,r) + \frac{1}{6}\partial F_3^{\pm}(t,r)y_3^{\pm}(t,r) + \frac{1}{6}\partial^3 F_3^{\pm}(t,r)y_1^{\pm}(t,r)^3 \\
& + \frac{1}{2}\partial^2 F_3^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r) + \frac{1}{24}\partial F_2^{\pm}(t,r)y_4^{\pm}(t,r) \\
& + \frac{1}{8}\partial^2 F_2^{\pm}(t,r)y_2^{\pm}(t,r)^2 + \frac{1}{6}\partial^2 F_2^{\pm} y_1(t,r)y_3^{\pm}(t,r) \\
& + \frac{1}{4}\partial^3 F_2^{\pm}(t,r)y_1^{\pm}(t,r)^2 y_2^{\pm}(t,r) + \frac{1}{24}\partial^4 F_2^{\pm}(t,r)y_1^{\pm}(t,r)^4 \\
& + \frac{1}{8}\partial^3 F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r)^2 + \frac{1}{12}\partial^4 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^3 y_2^{\pm}(t,r) \\
& + \frac{1}{120}\partial F_1^{\pm}(t,r)y_5(t,r) + \frac{1}{24}\partial^2 F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_4^{\pm}(t,r) \\
& + \frac{1}{120}\partial^4 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^5 + \frac{1}{12}\partial^2 F_1^{\pm}(t,r)y_2^{\pm}(t,r)y_3^{\pm}(t,r) \\
& + \partial^3 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2 y_3^{\pm}(t,r)]dt.
\end{aligned}
$$

Here $\partial^k F_l(s,r)$ means the $k-th$ partial derivative of the function $F_l(s,r)$ with respect to the variable r. Also from [14] we have the functions

$$
\begin{aligned}
f_1^{\pm}(r) &= \int_0^{\pm\pi} F_1^{\pm}(t,r)dt, \\
f_2^{\pm}(r) &= \int_0^{\pm\pi} [F_2^{\pm}(t,r) + \partial F_1^{\pm}(t,r)y_1^{\pm}(t,r)]dt, \\
f_3^{\pm}(r) &= \int_0^{\pm\pi} [F_3^{\pm}(t,r) + \partial F_2^{\pm}(t,r)y_1^{\pm}(t,r) \\
&\qquad + \frac{1}{2}\partial^2 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2 + \frac{1}{2}\partial F_1^{\pm}(t,r)y_2^{\pm}(t,r)]dt, \\
f_4^{\pm}(r) &= \int_0^{\pm\pi} [F_4^{\pm}(t,r) + \partial F_3^{\pm}(t,r)y_1^{\pm}(t,r) + \frac{1}{2}\partial^2 F_2^{\pm}(t,r)y_1^{\pm}(t,r)^2 \\
&\qquad + \frac{1}{2}\partial F_2^{\pm}(t,r)y_2^{\pm}(t,r) + \frac{1}{2}\partial^2 F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r) \\
&\qquad + \frac{1}{6}\partial^3 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^3 + \frac{1}{6}\partial F_1^{\pm}(t,r)y_3^{\pm}(t,r)]dt,
\end{aligned}
$$

$$\begin{aligned}
f_5^{\pm}(r) = \int_0^{\pm\pi} &[F_5^{\pm}(t,r) + \partial F_4^{\pm}(t,r)y_1^{\pm}(t,r) \\
&+ \frac{1}{2}\partial^2 F_3^{\pm}(t,r)y_1^{\pm}(t,r)^2 + \frac{1}{2}\partial F_3^{\pm}(t,r)y_2^{\pm}(t,r) \\
&+ \frac{1}{2}\partial^2 F_2^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r) + \frac{1}{6}\partial^3 F_2^{\pm}(t,r)y_1^{\pm}(t,r)^3 \\
&+ \frac{1}{6}\partial F_2^{\pm}(t,r)y_3^{\pm}(t,r) + \frac{1}{6}\partial^2 F_1^{\pm}(s,r)y_1^{\pm}(t,r)y_3^{\pm}(t,r) \\
&+ \frac{1}{8}\partial^2 F_1^{\pm}(t,r)y_2^{\pm}(t,r)^2 + \frac{1}{4}\partial^3 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2 y_2^{\pm}(t,r) \\
&+ \frac{1}{24}\partial^4 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^4 + \frac{1}{24}\partial F_1^{\pm}(t,r)y_4^{\pm}(t,r)]dt, \\
f_6^{\pm}(r) = \int_0^{\pm\pi} &[F_6^{\pm}(t,r) + \partial F_5^{\pm}(t,r)y_1^{\pm}(t,r) + \frac{1}{2}\partial^2 F_4^{\pm}(t,r)y_1^{\pm}(t,r)^2 \\
&+ \frac{1}{2}\partial F_4^{\pm}(t,r)y_2^{\pm}(t,r) + \frac{1}{6}\partial F_3^{\pm}(t,r)y_3^{\pm}(t,r) + \frac{1}{6}\partial^3 F_3^{\pm}(t,r)y_1^{\pm}(t,r)^3 \\
&+ \frac{1}{2}\partial^2 F_3^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r) + \frac{1}{24}\partial F_2^{\pm}(t,r)y_4^{\pm}(t,r) \\
&+ \frac{1}{8}\partial^2 F_2^{\pm}(t,r)y_2^{\pm}(t,r)^2 + \frac{1}{6}\partial^2 F_2^{\pm} y_1^{\pm}(t,r)y_3^{\pm}(t,r) \\
&+ \frac{1}{4}\partial^3 F_2^{\pm}(t,r)y_1^{\pm}(t,r)^2 y_2^{\pm}(t,r) + \frac{1}{24}\partial^4 F_2^{\pm}(t,r)y_1^{\pm}(t,r)^4 \\
&+ \frac{1}{8}\partial^3 F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r)^2 + \frac{1}{12}\partial^4 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^3 y_2^{\pm}(t,r) \\
&+ \frac{1}{120}\partial F_1^{\pm}(t,r)y_5^{\pm}(t,r) + \frac{1}{24}\partial^2 F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_4^{\pm}(t,r) \\
&+ \frac{1}{120}\partial^4 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^5 + \frac{1}{12}\partial^2 F_1^{\pm}(t,r)y_2^{\pm}(t,r)y_3^{\pm}(t,r) \\
&+ \partial^3 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2 y_3^{\pm}(t,r)]dt, \\
f_7^{\pm}(r) = \int_0^{\pm\pi} &[F_7^{\pm}(t,r) + \partial F_6^{\pm}(t,r)y_1^{\pm}(t,r) + \frac{1}{2}\partial F_5^{\pm}(t,r)y_2^{\pm}(t,r) \\
&+ \frac{1}{2}\partial^2 F_5^{\pm}(t,r)y_1^{\pm}(t,r)^2 + \frac{1}{6}\partial F_4^{\pm}(t,r)y_3(t,r) \\
&+ \frac{1}{2}\partial^2 F_4^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r) + \frac{1}{6}\partial^3 F_4^{\pm}(t,r)y_1^{\pm}(t,r)^3 \\
&+ \frac{1}{24}\partial F_3^{\pm}(t,r)y_4^{\pm}(t,r) + \frac{1}{6}\partial^2 F_3^{\pm}(t,r)y_1^{\pm}(t,r)y_3^{\pm}(t,r) \\
&+ \frac{1}{4}\partial^3 F_3^{\pm}(t,r)y_1^{\pm}(t,r)^2 y_2^{\pm}(t,r) + \frac{1}{24}\partial^4 F_3^{\pm}(t,r)y_1^{\pm}(t,r)^4 \\
&+ \frac{1}{8}\partial^2 F_3^{\pm}(t,r)y_2^{\pm}(t,r)^2 + \frac{1}{120}\partial F_2^{\pm}(t,r)y_5^{\pm}(t,r) \\
&+ \frac{1}{24}\partial^2 F_2^{\pm}(t,r)y_1(t,r)y_4^{\pm}(t,r) + \frac{1}{12}\partial^3 F_2^{\pm}(t,r)y_1^{\pm}(t,r)^2 y_3^{\pm}(t,r) \\
&+ \frac{1}{8}\partial^3 F_2^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r)^2 + \frac{1}{12}\partial^4 F_2^{\pm}(t,r)y_1^{\pm}(t,r)^3 y_2^{\pm}(t,r) \\
&+ \frac{1}{120}\partial^5 F_2^{\pm}(t,r)y_1^{\pm}(t,r)^5 + \frac{1}{12}\partial^2 F_2^{\pm}(t,r)y_2^{\pm}(t,r)y_3^{\pm}(t,r)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{720}\partial F_1^{\pm}(t,r)y_6^{\pm}(t,r)+\frac{1}{120}\partial^2 F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_5^{\pm}(t,r)\\
&+\frac{1}{48}\partial^3 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2y_4^{\pm}(t,r)+\frac{1}{48}\partial^2 F_1^{\pm}(t,r)y_2^{\pm}(t,r)y_4^{\pm}(t,r)\\
&+\frac{1}{36}\partial^4 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^3y_3^{\pm}(t,r)+\frac{1}{72}\partial^2 F_1^{\pm}(t,r)y_3^{\pm}(t,r)^2\\
&+\frac{1}{48}\partial^5 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^4y_2^{\pm}(t,r)+\frac{1}{16}\partial^4 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2y_2^{\pm}(t,r)^2\\
&+\frac{1}{48}\partial^3 F_1^{\pm}(t,r)y_2^{\pm}(t,r)^3+\frac{1}{12}\partial^3 F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r)y_3^{\pm}(t,r)\\
&+\frac{1}{720}\partial^6 F_1^{\pm}(t,r)]dt.
\end{aligned}
$$

The function $f_k(r)=f_k^+(r)-f_k^-(r)$ is called the averaged function of order k. If $f_\ell(r)\equiv 0$ for $\ell\in\{1,\dots,6\}$ but $f_{\ell+1}(r)\not\equiv 0$, then the simple positive real roots of the functions $f_{\ell+1}(r)$ provide limit cycles of the piecewise differential system (3).

3. Proof of Theorem 1

Consider the linear center we shall study which periodic orbits of this center become limit cycles when we perturb the center inside the discontinuous piecewise differential systems formed by systems (1) and (2), i.e. in $y\geq 0$ we have the differential system

$$
\begin{aligned}
\dot{x}&=-y+\alpha x+\beta y+\gamma,\\
\dot{y}&=x-\beta x+\alpha y+\delta,
\end{aligned}
$$

and in $y\leq 0$ we have the differential system

$$
\begin{aligned}
\dot{x}&=-y-bx^2-cxy-dy^2,\\
\dot{y}&=x+ax^2+Axy-ay^2,
\end{aligned}
$$

where

$$
\begin{aligned}
a=&a_1\varepsilon+a_2\varepsilon^2+a_3\varepsilon^3+a_4\varepsilon^4+a_5\varepsilon^5+a_6\varepsilon^6+a_7\varepsilon^7,\\
b=&b_1\varepsilon+b_2\varepsilon^2+b_3\varepsilon^3+b_4\varepsilon^4+b_5\varepsilon^5+b_6\varepsilon^6+b_7\varepsilon^7,\\
c=&c_1\varepsilon+c_2\varepsilon^2+c_3\varepsilon^3+c_4\varepsilon^4+c_5\varepsilon^5+c_6\varepsilon^6+c_7\varepsilon^7,\\
d=&d_1\varepsilon+d_2\varepsilon^2+d_3\varepsilon^3+d_4\varepsilon^4+d_5\varepsilon^5+d_6\varepsilon^6+d_7\varepsilon^7,\\
A=&A_1\varepsilon+A_2\varepsilon^2+A_3\varepsilon^3+A_4\varepsilon^4+A_5\varepsilon^5+A_6\varepsilon^6+A_7\varepsilon^7,\\
\alpha=&\alpha_1\varepsilon+\alpha_2\varepsilon^2+\alpha_3\varepsilon^3+\alpha_4\varepsilon^4+\alpha_5\varepsilon^5+\alpha_6\varepsilon^6+\alpha_7\varepsilon^7,\\
\beta=&-1+\beta_1\varepsilon+\beta_2\varepsilon^2+\beta_3\varepsilon^3+\beta_4\varepsilon^4+\beta_5\varepsilon^5+\beta_6\varepsilon^6+\beta_7\varepsilon^7,\\
\gamma=&\gamma_1\varepsilon+\gamma_2\varepsilon^2+\gamma_3\varepsilon^3+\gamma_4\varepsilon^4+\gamma_5\varepsilon^5+\gamma_6\varepsilon^6+\gamma_7\varepsilon^7,\ \text{and}\\
\delta=&\delta_1\varepsilon+\delta_2\varepsilon^2+\delta_3\varepsilon^3+\delta_4\varepsilon^4+\delta_5\varepsilon^5+\delta_6\varepsilon^6+\delta_7\varepsilon^7.
\end{aligned}
$$

We have developed the parameters of the differential systems until seven order in ε, because then each parameter can contribute in all the averaged functions until order seven, otherwise the results obtained will be more poor with respect to the number of limit cycles that the piecewise differential systems here studied can exhibit. Moreover in the expression of β the -1 is there because we want that when $\varepsilon=0$ the linear differential system (1) has a center.

After we write the discontinuous piecewise differential system in polar coordinates $(\dot{r},\dot{\theta})$, where $x=r\cos(\theta)$ and $y=r\sin(\theta)$. Then we take as independent variable the angle θ, and the system $(\dot{r},\dot{\theta})$ becomes the differential equation $dr/d\theta$. By doing a Taylor expansion truncated at 7-th order in ε we obtain an expression for $dr/d\theta$ written as the one of the differential system (3). In short we have written our discontinuous piecewise differential system formed by systems (1) and (2) in the normal form (3) for applying the averaging theory. We give only the expression of functions $F_i^{\pm}(r,\theta)$ for $i=1,...,3$. The explicit expressions of $F_i^{\pm}(r,\theta)$ for $i=4,5,6,7$ are quite large so we omit them, but they can be obtained easily using an algebraic manipulator as mathematica or mapple,

$$
\begin{aligned}
F_1^+(r,\theta) =& \gamma_1\cos(\theta)+\delta_1\sin(\theta)+\alpha_1 r,\\
F_1^-(r,\theta) =& -r^2\left((c_1-a_1)\sin(\theta)\cos^2(\theta)+a_1\sin^3(\theta)+(d_1-A_1)\sin^2(\theta)\cos(\theta)+b_1\cos^3(\theta)\right),\\
F_2^+(r,\theta) =& \left(-2\gamma_1\delta_1\cos(2\theta)+\gamma_1^2\sin(2\theta)-\delta_1^2\sin(2\theta)+2\alpha_1\beta_1 r^2+2\alpha_2 r^2+2\alpha_1\gamma_1 r\sin(\theta)\right.\\
&\left.-2\alpha_1\delta_1 r\cos(\theta)+2\beta_1\gamma_1 r\cos(\theta)+2\beta_1\delta_1 r\sin(\theta)+2\gamma_2 r\cos(\theta)+2\delta_2 r\sin(\theta)\right)/(2r),\\
F_2^-(r,\theta) =& -\frac{1}{8}\sin(2\theta)a_1^2r^3-\frac{1}{8}\sin(6\theta)a_1^2r^3-\frac{3}{32}\sin(2\theta)A_1^2r^3+\frac{1}{32}\sin(6\theta)A_1^2r^3+\frac{5}{32}\sin(2\theta)b_1^2r^3\\
&+\frac{1}{8}\sin(4\theta)b_1^2r^3+\frac{1}{32}\sin(6\theta)b_1^2r^3+\frac{3}{32}\sin(2\theta)c_1^2r^3-\frac{1}{32}\sin(6\theta)c_1^2r^3+\frac{5}{32}\sin(2\theta)d_1^2r^3\\
&-\frac{1}{8}\sin(4\theta)d_1^2r^3+\frac{1}{32}\sin(6\theta)d_1^2r^3-\frac{1}{8}\cos(2\theta)a_1A_1r^3+\frac{1}{8}\cos(6\theta)a_1A_1r^3+\frac{1}{4}a_1b_1r^3\\
&+\frac{3}{8}\cos(2\theta)a_1b_1r^3+\frac{1}{4}\cos(4\theta)a_1b_1r^3+\frac{1}{8}\cos(6\theta)a_1b_1r^3+\frac{1}{16}\sin(2\theta)A_1b_1r^3\\
&+\frac{1}{8}\sin(4\theta)A_1b_1r^3+\frac{1}{16}\sin(6\theta)A_1b_1r^3+\frac{1}{8}\sin(2\theta)a_1c_1r^3+\frac{1}{8}\sin(6\theta)a_1c_1r^3\\
&+\frac{1}{16}\cos(2\theta)A_1c_1r^3-\frac{1}{16}\cos(6\theta)A_1c_1r^3+\frac{1}{8}b_1c_1r^3+\frac{1}{16}\cos(2\theta)b_1c_1r^3-\frac{1}{8}\cos(4\theta)b_1c_1r^3\\
&-\frac{1}{16}\cos(6\theta)b_1c_1r^3+\frac{1}{4}a_1d_1r^3-\frac{3}{8}\cos(2\theta)a_1d_1r^3+\frac{1}{4}\cos(4\theta)a_1d_1r^3-\frac{1}{8}\cos(6\theta)a_1d_1r^3\\
&-\frac{1}{16}\sin(2\theta)A_1d_1r^3+\frac{1}{8}\sin(4\theta)A_1d_1r^3-\frac{1}{16}\sin(6\theta)A_1d_1r^3+\frac{3}{16}\sin(2\theta)b_1d_1r^3\\
&-\frac{1}{16}\sin(6\theta)b_1d_1r^3+\frac{1}{8}c_1d_1r^3-\frac{1}{16}\cos(2\theta)c_1d_1r^3-\frac{1}{8}\cos(4\theta)c_1d_1r^3+\frac{1}{16}\cos(6\theta)c_1d_1r^3\\
&-\frac{1}{2}\sin(\theta)a_2r^2+\frac{1}{2}\sin(3\theta)a_2r^2+\frac{1}{4}\cos(\theta)A_2r^2-\frac{1}{4}\cos(3\theta)A_2r^2-\frac{3}{4}\cos(\theta)b_2r^2\\
&-\frac{1}{4}\cos(3\theta)b_2r^2-\frac{1}{4}\sin(\theta)c_2r^2-\frac{1}{4}\sin(3\theta)c_2r^2-\frac{1}{4}\cos(\theta)d_2r^2+\frac{1}{4}\cos(3\theta)d_2r^2,
\end{aligned}
$$

$$
\begin{aligned}
F_3^+(r,\theta) =& \gamma_3\cos(\theta)+\frac{(\gamma_2\cos(\theta)+\delta_2\sin(\theta)+\alpha_2 r)(\gamma_1\sin(\theta)-\delta_1\cos(\theta)+\beta_1 r)}{r}+\alpha_3 r\\
&+\delta_3\sin(\theta)+\left((\gamma_1\cos(\theta)+\delta_1\sin(\theta)+\alpha_1 r)\left(\gamma_1^2\sin^2(\theta)+\delta_1^2\cos^2(\theta)+\left(\beta_1^2+\beta_2\right)r^2\right.\right.\\
&\left.\left.-\cos(\theta)\left(2\gamma_1\delta_1\sin(\theta)+r\left(2\beta_1\delta_1+\delta_2\right)\right)+r\left(2\beta_1\gamma_1+\gamma_2\right)\sin(\theta)\right)\right)/r^2,\\
F_3^-(r,\theta) =& r^2\left(r\left(-(a_1-c_1)\sin^2(\theta)\cos(\theta)+a_1\cos^3(\theta)+(A_1+b_1)\sin(\theta)\cos^2(\theta)\right.\right.\\
&\left.+d_1\sin^3(\theta)\right)\left(-(a_2-c_2)\sin(\theta)\cos^2(\theta)+a_2\sin^3(\theta)-(A_2-d_2)\sin^2(\theta)\cos(\theta)+b_2\cos^3(\theta)\right)\\
&-r\left(-(a_1-c_1)\sin(\theta)\cos^2(\theta)+a_1\sin^3(\theta)-(A_1-d_1)\sin^2(\theta)\cos(\theta)+b_1\cos^3(\theta)\right)\\
&\left(r\left(-(a_1-c_1)\sin^2(\theta)\cos(\theta)+a_1\cos^3(\theta)+(A_1+b_1)\sin(\theta)\cos^2(\theta)+d_1\sin^3(\theta)\right)^2\right.\\
&-a_2\cos^3(\theta)+a_2\sin^2(\theta)\cos(\theta)+A_2(-\sin(\theta))\cos^2(\theta)-b_2\sin(\theta)\cos^2(\theta)-c_2\sin^2(\theta)\cos(\theta)\\
&\left.\left.-d_2\sin^3(\theta)\right)+(a_3-c_3)\sin(\theta)\cos^2(\theta)-a_3\sin^3(\theta)+(A_3-d_3)\sin^2(\theta)\cos(\theta)-b_3\cos^3(\theta)\right).
\end{aligned}
$$

Now we compute the averaged function $f_i(r)$ defined in section 2, and for $i=1$ we get

$$f_1(r)=\frac{2}{3}\left(a_1+c_1\right)r^2+\pi\alpha_1 r+2\delta_1.$$

So the polynomial $f_1(r)$ can have at most two positive real roots r_1 and r_2, which provide two limit cycles for the discontinuous piecewise differential system (1)-(2) when ε is sufficiently small. These limit cycles tend to the circular periodic orbits of radius r_1 and r_2 of the linear differential center $\dot{x} = -y$, $\dot{y} = x$ when $\varepsilon \to 0$.

In order to apply the averaging theory of second order we need that $f_1(r) \equiv 0$. In order to eliminate the coefficients of this polynomial we must take $c_1 = -a_1$, $\alpha_1 = 0$ and $\delta_1 = 0$. Computing the function $f_2(r)$ we obtain

$$f_2(r) = \frac{\pi}{8} a_1 (b_1 + d_1) r^3 + \frac{2}{3} (a_2 + c_2) r^2 + \pi \alpha_2 r + 2\delta_2.$$

This polynomial can have at most three positive real roots, and consequently the averaging theory up to order 2 can provide at most three limit cycles for the discontinuous piecewise differential system (1)-(2) when ε is sufficiently small, which again when $\varepsilon \to 0$ they will tend to the circular periodic orbits of the linear differential center $\dot{x} = -y$, $\dot{y} = x$ of radius the roots of the polynomial $f_2(r)$.

In order to apply the averaging theory of third order we need to have $f_2(r) \equiv 0$, for that we must take $c_2 = -a_2$, $\alpha_2 = 0$ and $\delta_2 = 0$ in order to eliminate the coefficients of r^2, r and the constant term. For the coefficient of r^3 we have two cases $b_1 = -d_1$ or $a_1 = 0$. Therefore we start with the first case $b_1 = -d_1$.

Case 1: $b_1 = -d_1$ **and** $a_1 \neq 0$**.** Computing the function $f_3(r)$ we obtain

$$f_3(r) = \frac{2}{5} a_1 b_1^2 r^4 + \frac{\pi}{8} a_1 (b_2 + d_2) r^3 + \frac{2}{3} (a_3 + c_3) r^2 + \pi \alpha_3 r + 2\delta_3.$$

Then the polynomial $f_3(r)$ can have at most four positive real roots, and therefore provide when ε is sufficiently small at most four limit cycles for the discontinuous piecewise differential system (1)-(2).

In order to apply the averaging theory of fourth order we need that $f_3(r) \equiv 0$. So we must take

$$b_1 = 0, \ d_2 = -b_2, \ c_3 = -a_3, \ \alpha_3 = 0 \text{ and } \delta_3 = 0.$$

Then computing the function $f_4(r)$ we get

$$f_4(r) = \frac{\pi}{8} a_1 (b_3 + d_3) r^3 + \frac{2}{3} (a_4 + c_4) r^2 + \pi \alpha_4 r + 2\delta_4.$$

So the polynomial $f_4(r)$ can have at most three positive real roots, and produce at most three limit cycles for the discontinuous piecewise differential system (1)-(2) when ε is suffiiciently small.

In order to apply the averaging theory of fifth order we need that $f_4(r) \equiv 0$, for that we must take

$$d_3 = -b_3, \ c_4 = -a_4, \ \alpha_4 = 0 \text{ and } \delta_4 = 0.$$

Computing the function $f_5(r)$ we obtain

$$f_5(r) = \frac{2}{5} a_1 b_2^2 r^4 + \frac{\pi}{8} a_1 (b_4 + d_4) r^3 + \frac{2}{3} (a_5 + c_5) r^2 + \pi \alpha_5 r + 2\delta_5.$$

So the polynomial $f_5(r)$ can have at most four positive real roots, and consequently the discontinuous piecewise differential system (1)-(2) can have at most four limit cycles for ε sufficiently small.

In order to apply the averaging theory of sixth order we need that $f_5(r) \equiv 0$, therefore it is necessary to take

$$b_2 = 0, \ d_4 = -b_4, \ c_5 = -a_5, \ \alpha_5 = 0 \text{ and } \delta_5 = 0.$$

Computing the function $f_6(r)$ we get

$$f_6(r) = \frac{\pi}{8} a_1 (b_5 + d_5) r^3 + \frac{2}{3} (a_6 + c_6) r^2 + \pi \alpha_6 r + 2\delta_6$$

This polynomial can have at most three positive real roots.

In order to apply the averaging theory of seventh order we must have $f_6(r) \equiv 0$. So in order to eliminate the coefficients of $f_6(r)$ we must take

$$d_5 = -b_5,\ c_6 = -a_6,\ \alpha_6 = 0,\ \text{and}\ \delta_6 = 0.$$

Computing the function $f_7(r)$ we get

$$f_7(r) = \quad Ar^8 + Br^7 + Cr^6 + Dr^5 + \tfrac{2}{5}a_1 b_3^2 r^4 + \tfrac{1}{8}\pi a_1 \left(b_6 + d_6\right) r^3 + \tfrac{2}{3}\left(a_7 + c_7\right) r^2 + \pi\alpha_7 r + 2\delta_7,$$

such that

$$A = -\frac{128a_1^3\left(-57258a_1^2A_1^2 + 904365a_1^4 - 210418A_1^4\right)}{5892561675},$$

$$B = -\frac{\left(467937a_1^5A_2 - 733185a_2a_1^4A_1 - 83718a_1^3A_1^2A_2 - 232218a_2a_1^2A_1^3 + 385a_1A_1^4A_2 - 385a_2A_1^5\right)\pi}{39813120},$$

$$\begin{aligned} C = \quad & -\frac{64}{18243225}\left(-8811a_1^3A_1b_3 + 120a_1A_1^3b_3 + 4824a_1^3A_2^2 - 1080a_1^3A_1A_3 + 10824a_3a_1^2A_1^2\right.\\ & -25608a_2a_1^2A_1A_2 - 144a_2^2a_1A_1^2 - 720a_1A_1^2A_2^2 - 3280a_1A_1^3A_3 + 3280a_3A_1^4 + 720a_2A_1^3A_2\\ & \left.+10962a_3a_1^4 - 23544a_2^2a_1^3\right), \end{aligned}$$

$$\begin{aligned} D = \quad & \left(1820a_1A_1A_2b_3 - 336a_1A_1^2b_4 - 770a_2A_1^2b_3 - 504a_1^3A_4 + 504a_4a_1^2A_1 - 1365a_3a_1^2A_2 + 504a_4A_1^3\right.\\ & +1575a_2a_1^2A_3 + 210a_1A_2^3 - 210a_2a_3a_1A_1 + 210a_2^2a_1A_2 + 210a_1A_1A_2A_3 - 504a_1A_1^2A_4\\ & \left.-210a_2A_1A_2^2 - 210a_2^3A_1 - 1575a_3A_1^2A_2 + 1365a_2A_1^2A_3 - 144a_1^3b_4 + 450a_2a_1^2b_3\right)\pi/69120. \end{aligned}$$

This polynomial can have at most eight positive real roots.

Now we continue the computations just after $f_2(r)$ taking the second case $a_1 = 0$.

Case 2: $a_1 = 0$**.** Computing the function $f_3(r)$ we obtain

$$f_3(r) = \frac{\pi}{8}a_2\left(b_1 + d_1\right)r^3 + \frac{2}{3}\left(a_3 + c_3\right)r^2 + \pi\alpha_3 r + 2\delta_3.$$

This polynomial can have at most three positive real roots. In order to apply the averaging theory of fourth order we must have $f_3(r) \equiv 0$. So we must take $c_3 = -a_3,\ \alpha_3 = 0,\ \delta_3 = 0$, and in order to eliminate the coefficient of r^3 in f_3 we must take $d_1 = -b_1$ or $a_2 = 0$. Then there are two subcases.

Subcase* 2.1*: $d_1 = -b_1$ ***and*** $a_2 \neq 0$**.** Computing $f_4(r)$ we get

$$f_4(r) = \frac{2}{5}a_2b_1^2r^4 + \frac{\pi}{8}a_2\left(b_2 + d_2\right)r^3 + \frac{2}{3}\left(a_4 + c_4\right)r^2 + \pi\alpha_4 r + 2\delta_4.$$

This polynomial can have at most four positive real roots. In order to apply the averaging theory of fifth order we must have $f_4(r) \equiv 0$. Therefore we need to $b_1 = 0,\ c_4 = -a_4,\ d_2 = -b_2,\ \alpha_4 = 0$ and $\delta_4 = 0$. Computing $f_5(r)$ we get

$$f_5(r) = \frac{\pi}{8}a_2\left(b_3 + d_3\right)r^3 + \frac{2}{3}\left(a_5 + c_5\right)r^2 + \pi\alpha_5 r + 2\delta_5.$$

This polynomial can have at most three positive real roots. In order to apply the averaging theory of sixth order we must have $f_5(r) \equiv 0$. So we must take $d_3 = -b_3,\ c_5 = -a_5,\ \alpha_5 = 0$ and $\delta_5 = 0$. Computing $f_6(r)$ we get

$$f_6(r) = \frac{2}{5}a_2b_2^2r^4 + \frac{\pi}{8}a_2\left(b_4 + d_4\right)r^3 + \frac{2}{3}\left(a_6 + c_6\right)r^2 + \pi\alpha_6 r + 2\delta_6.$$

This polynomial can have at most five positive real roots. In order to apply the averaging theory of seventh order we need to have $f_6(r) \equiv 0$. So we must take $b_2 = 0,\ d_4 = -b_4,\ c_6 = -d_6,\ \alpha_6 = 0$ and $\delta_6 = 0$. Computing $f_7(r)$ we have

$$f_7(r) = \frac{77\pi a_2A_1^5}{7962624}r^7 - \frac{1024A_1^3\left(41a_3A_1 + 9a_2A_2\right)}{3648645}r^6 + Er^5 + \frac{\pi}{8}a_2\left(b_5 + d_5\right)r^3 + \frac{2}{3}\left(a_7 + c_7\right)r^2 + \pi\alpha_7 r + 2\delta_7,$$

such that

$$E = \frac{7\pi A_1 \left(-110 a_2 A_1 b_3 - 30 a_2 A_2^2 + 195 a_2 A_1 A_3 + 72 a_4 A_1^2 - 225 a_3 A_1 A_2 - 30 a_2^3\right)}{69120}.$$

This polynomial can have at most six positive real roots by the Descartes Theorem, which states: *Consider the real polynomial* $p(x) = a_{i_1} x^{i_1} + a_{i_2} x^{i_2} + \cdots + a_{i_r} x^{i_r}$ *with* $0 \le i_1 < i_2 < \cdots < i_r$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in \{1, 2, \cdots, r\}$. *When* $a_{i_j} a_{i_{j+1}} < 0$ *we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* m, *then* $p(x)$ *has at most* m *positive real roots. Moreover it is always possible to choose the coefficients of* $p(x)$ *in such a way that* $p(x)$ *has exactly* $r-1$ *positive real roots.* For a proof see [3].

Now we start the computations from $f_3(r)$ just before the Subcase 2.1 taking the second subcase $a_2 = 0$.

Subcase **2.2*:*** $a_2 = 0$. Therefore we take $a_2 = 0,\ c_3 = -a_3,\ \alpha_3 = 0$ and $\delta_3 = 0$; which give $f_3(r) = 0$. Now computing $f_4(r)$ we get

$$f_4(r) = \frac{\pi}{8} a_3 \left(b_1 + d_1\right) r^3 + \frac{2}{3} \left(a_4 + c_4\right) r^2 + \pi \alpha_4 r + 2\delta_4.$$

This polynomial can have at most three positive real roots. In order to apply the averaging theory of fifth order we must have $f_4(r) \equiv 0$. So we need $c_4 = -a_4,\ \alpha_4 = 0,\ \delta_4 = 0$, and in order to eliminate the coefficient of r^3 we need to have $d_1 = -b_1$ or $a_3 = 0$. Here also we have two subcases.

Subcase **2.2.1*:*** $d_1 = -b_1$ ***and*** $a_3 \neq 0$***.*** Computing $f_5(r)$ we get

$$f_5(r) = \frac{2}{5} a_3 b_1^2 r^4 + \frac{\pi}{8} a_3 (b_2 + d_2) r^3 + \frac{2}{3} \left(a_5 + c_5\right) r^2 + \pi \alpha_5 r + 2\delta_5.$$

This polynomial can have at most four positive real roots. In order to apply the averaging theory of sixth order we should have $f_5(r) \equiv 0$. So we need $b_1 = 0,\ d_2 = -b_2,\ c_5 = -a_5,\ \alpha_5 = 0$ and $\delta_5 = 0$. Now we compute $f_6(r)$ and we obtain

$$f_6(r) = \frac{\pi}{8} a_3 \left(b_3 + d_3\right) r^3 + \frac{2}{3} \left(a_6 + c_6\right) r^2 + \pi \alpha_6 r + 2\delta_6.$$

This polynomial can have at most three positive real roots. In order to apply the averaging theory of seventh order we must have $f_6(r) \equiv 0$. So we must take $d_3 = -b_3,\ c_6 = -a_6,\ \alpha_6 = 0$ and $\delta_6 = 0$. Computing $f_7(r)$ we get

$$\begin{aligned} f_7(r) = \quad & -\tfrac{41984 a_3 A_1^4}{3648645} r^6 + \tfrac{7\pi A_1^2 (72 a_4 A_1 - 225 a_3 A_2 + 170 a_3 b_2)}{69120} r^5 + \tfrac{2}{5} a_3 b_2^2 r^4 + \tfrac{\pi}{8} a_3 \left(b_4 + d_4\right) r^3 \\ & + \tfrac{2}{3} \left(a_7 + c_7\right) r^2 + \pi \alpha_7 r + 2\delta_7. \end{aligned}$$

This polynomial can have at most six positive real roots.

Subcase **2.2.2*:*** $a_3 = 0$. Hence we return to the fourth order and we take $c_4 = -a_4,\ \alpha_4 = 0,\ \delta_4 = 0$, which give $f_4(r) \equiv 0$. Computing $f_5(r)$ we get

$$f_5(r) = \frac{\pi}{8} a_4 (b_1 + d_1) r^3 + \frac{2}{3} (a_5 + c_5) r^2 + \pi \alpha_5 r + 2\delta_5.$$

This polynomial can have at most three positive real roots. In order to apply the averaging theory of sixth order we must have $f_5(r) \equiv 0$. So we must take $c_5 = -a_5,\ \alpha_5 = 0,\ \delta_5 = 0$, and in order to eliminate the coefficient of r^3 we have two subcases $d_1 = -b_1$ or $a_4 = 0$.

Subcase* 2.2.2.1*: $d_1 = -b_1$ ***and*** $a_4 \neq 0$. Computing $f_6(r)$ in this case we get

$$f_6(r) = \frac{2}{5}a_4 d_1^2 r^4 + \frac{\pi}{8}a_4(b_2 + d_2)r^3 + \frac{2}{3}(a_6 + c_6)r^2 + \pi\alpha_6 r + 2\delta_6.$$

This polynomial can have at most four positive real roots. In order to apply the averaging theory of seventh order we must have $f_6(r) \equiv 0$. So we must take $d_1 = 0,\ d_2 = -b_2,\ c_6 = -a_6,\ \alpha_6 = 0$, and $\delta_6 = 0$. Computing $f_7(r)$ we get

$$f_7(r) = \frac{7\pi}{960}a_4 A_1^3 r^5 + \frac{\pi}{8}a_4\left(b_3 + d_3\right)r^3 + \frac{2}{3}\left(a_7 + c_7\right)r^2 + \pi\alpha_7 r + 2\delta_7$$

This polynomial can have at most four positive real roots by the Descartes Theorem.

Subcase* 2.2.2.2*: $a_4 \neq 0$. We return to the fifth order and we take $a_4 = 0,\ c_5 = -a_5,\ \alpha_5 = 0,\ \delta_5 = 0$, which give $f_5(r) \equiv 0$. Computing $f_6(r)$ we get

$$f_6(r) = -\frac{\pi}{8}c_5(b_1 + d_1)r^3 + \frac{2}{3}(a_6 + c_6)r^2 + \pi\alpha_6 r + 2\delta_6.$$

This polynomial can have at most three positive real roots. In order to apply the averaging theory of seventh order we must have $f_6(r) \equiv 0$. So we must take $c_6 = -a_6,\ \alpha_6 = 0,\ \delta_6 = 0$, and in order to eliminate the coefficient of r^3 here also we have two cases $d_1 = -b_1$ or $c_5 = 0$. For $d_1 = -b_1$ computing $f_7(r)$ we obtain

$$f_7(r) = -\frac{2}{5}b_1^2 c_5 r^4 - \frac{\pi}{8}c_5\left(b_2 + d_2\right)r^3 + \frac{2}{3}\left(a_7 + c_7\right)r^2 + \pi\alpha_7 r + 2\delta_7$$

This polynomial can have at most four positive real roots. Computing $f_7(r)$ in the case $c_5 = 0$ we get

$$f_7(r) = -\frac{\pi}{8}c_6(b_1 + d_1)r^3 + \frac{2}{3}(a_7 + c_7)r^2 + \pi\alpha_7 r + 2\delta_7.$$

This polynomial can have at most three positive real roots.

In summary, in all the previous cases the polynomials $f_i(r)$ can have at most 2, 3, 4, 6 and 8 real positive roots. Hence the maximum number of limit cycles that we can obtain using the averaging theory up to seven order is eight. This completes the proof of Theorem 1.

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References

[1] A. A. Andronov, A. A. Vitt, and S. E. Khaikin, *Theory of Oscillators*, International Series of Monographs in Physics, volume 4. Oxford, 1966.
[2] S. Banerjee and G. C. Verghese, *Nonlinear phenomena in power electronics*, IEEE, 1999.
[3] I. Berezin and N. Zhidkov, *Computing methods*, Pergamon Press, Oxford, Volume II, 1964.
[4] C. Buzzi, C. Pessoa and J. Torregrosa, *Piecewise linear perturbations of a linear center*, Discrete Contin. Dyn. Syst. **9** (2013), 3915–3936.
[5] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative Theory of Planar Differential Systems*, UniversiText, Springer–Verlag, New York, 2006.
[6] A. Filippov, *Differential Equation with Discontinuous Right-Hand Sides*, Kluwer Academic, Amsterdam, 1988.
[7] J. Giné, *On some open problems in planar differential systems and Hilbert´s 16th problem*, Chaos, Solitons Fractals, 31(5) (2007), 1118–1134.
[8] M. Di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise-Smooth Dynamical Systems: Theory and Applications*, Applied Math. Sciences series **163**, Springer, London, 2008.
[9] E. Freire, E. Ponce and F. Torres, *A general mechanism to generate three limit cycles in planar Filippov systems with two zones*, Nonlinear Dyn. **78** (2014), 251–263.

[10] F. Giannakopoulos and K. Pliete, *Planar systems of piecewise linear differential equations with a line of discontinuity*, Nonlinearity **14** (2001), 1611–1632.
[11] S.M. Huan and X.S. Yang, *On the number of limit cycles in general planar piecewise linear systems*, Discrete Contin. Dyn. Syst. Ser. A **32** (2012), 2147–2164.
[12] A. Isidori, *Nonlinear Control Systems*, Springer, London, 1996.
[13] J. Itikawa, J. Llibre, A. C. Mereu, and R. Oliveira, *Limit cycles in uniform isochronous centers of discontinuous differential systems with four zones. Discrete*, Continuous Dynamical Systems-B, **22**(9) (2017), 3259–3272.
[14] J. Itikawa, J. Llibre, and D. D. Novaes, *A new result on averaging theory for a class of discontinuous planar differential systems with applications*, Rev. Mat. Iberoam. **33**(4) (2017), 1247–1265.
[15] O. Katsuhiko, *Modern Control Engineering*, 2nd edn. Prentice-Hall, Upper Saddle River, 1990.
[16] S. Li, C. Liu and J. Llibre, *The planar discontinuous piecewise linear refracting systems have at most one limit cycle*, Nonlinear Anal. Hybrid Syst. **41** 101045, (2021). 14 pp.
[17] L. Li, *Three crossing limit cycles in planar piecewise linear systems with saddle-focus type*, Electron. J. Qual. Theory Differ. Equ. **70** (2014), 1–14.
[18] J. Llibre, D. Novaes and M.A. Teixeira, *Maximum number of limit cycles for certain piecewise linear dynamical systems*, Nonlinear Dynamics **82** (2015), 1159–1175.
[19] J. Llibre, M. Ordóñez and E. Ponce, *On the existence and uniqueness of limit cycles in a planar piecewise linear systems without symmetry*, Nonlinear Anal. Ser. B Real World Appl. **14** (2013), 2002–2012.
[20] J. Llibre and E. Ponce, *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones*, Dyn. Contin. Discrete Impul. Syst. Ser. B **19** (2012), 325–335.
[21] J. Llibre and M.A. Teixeira, *Piecewise linear differential systems with only centers can create limit cycles*?, Nonlinear Dynam. **91** (2018), 249–255.
[22] J. Llibre and C. Valls, *Limit cycles of piecewise differential systems with only linear Hamiltonian saddles*, Symmetry **13** (2021), 1128, 10 pp.
[23] J. Llibre and X. Zhang, *Limit cycles created by piecewise linear centers*, Chaos **29** (2019), 053116, 14 pp.
[24] O. Makarenkov and J.S.W. Lamb, *Dynamics and bifurcations of nonsmooth systems: a survey*, Phys. D **241** (2012), 1826–1844.
[25] R. Ortega, *Periodic perturbations of an isochronous center*, Qualitative Theory of Dynamical Systems **3**(1) (2002), 83–91.
[26] D.J.W. Simpson, *Bifurcations in Piecewise-Smooth Continuous Systems*, World Scientific Series on Nonlinear Science A, vol. **69**, World Scientific, Singapore, 2010.
[27] M.A. Teixeira, *Perturbation theory for non-smooth systems*, Mathematics of Complexity and Dynamical Systems, vol. **1-3**, Springer, New York, 2012, 1325–1336 pp.
[28] Z. T. Zhusubaliyev and E. Mosekilde, *Bifurcations and chaos in piecewise-smooth dynamical systems*, vol. **44**, World Scientific, 2003.

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