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The Geometry of Quadratic Differential Systems with a Weak Focus of Third Order

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Abstract. In this article we determine the global geometry of the planar quadratic differential systems with a weak focus of third order. This class plays a significant role in the context of Hilbert's 16-th problem. Indeed, all examples of quadratic differential systems with at least four limit cycles, were obtained by perturbing a system in this family. We use the algebro-geometric concepts of divisor and zero-cycle to encode global properties of the systems and to give structure to this class. We give a theorem of topological classification of such systems in terms of integer-valued affine invariants. According to the possible values taken by them in this family we obtain a total of 18 topologically distinct phase portraits. We show that inside the class of all quadratic systems with the topology of the coefficients, there exists a neighborhood of the family of quadratic systems with a weak focus of third order and which may have graphics but no polycycle in the sense of [15] and no limit cycle, such that any quadratic system in this neighborhood has at most four limit cycles.

1 Introduction, Brief Review of the Literature and Informal Outline of Results

The complete characterization of the phase portraits for real planar quadratic vector fields is not known, and attempting to classify these systems, which occur rather often in applications, is quite a complex task. This family of systems depends on twelve parameters, but due to the group action of real affine transformations and positive time rescaling, the class ultimately depends on five parameters. Bifurcation diagrams were constructed for some algebraic and semialgebraic subsets of this class, see for example [4, 5, 8, 10, 25, 33, 41, 39, 50, 54]. With the exception of some articles (for example [50]) the classifications of systems were done in terms of local charts and inequalities on the coefficients of the systems written in these charts. This line of work follows the program stated by Coppel in his nice short article which appeared in 1966 (*cf.* [11]). Coppel thought that the phase portraits of quadratic systems could be characterized by means of algebraic inequalities on the coefficients. We now know that algebraic inequalities would not be sufficient; analytic as well as non-analytic ones (*cf.* references [37] and [14, pp. 118–119]) need to be taken into consideration. We would also want to see that classifications be done not in terms of coordinate charts as in previous works but in more intrinsic terms, to reveal the geometry of the

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