A Note on a Conjecture of Smale

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Introduction. The full classification of the invariant manifolds I_{hc} of the planar three-body problem (see [5], [2], [3], [1]) for positive masses of the bodies takes into account the order of the values of the normalized potential $\bar{V}$ at the critical points. Let L be a Lagrangian point and E_{ijk} the Eulerian one corresponding to the bodies on a line in the order i, j, k. In [5] (see also [1] page 740) Smale conjectured that for almost all choices of m_1, m_2, m_3 the numbers $\bar{V}(L)$, $\bar{V}(E_{123})$, $\bar{V}(E_{231})$, $\bar{V}(E_{312})$ are distinct. In this note we prove the conjecture and we point out that for a given order of the masses, i.e. $m_1 \geqslant m_2 \geqslant m_3$, we have a related order of the potentials $\bar{V}(L) > \bar{V}(E_{312}) \geqslant \bar{V}(E_{123}) \geqslant \bar{V}(E_{231})$. In a neighbourhood of the limiting cases we prove the inequalities analytically, and in the remaining region we show the results of a numerical computation.

In the planar three-body problem with normalized masses $\sum_{i=1}^{3} m_i = 1$ the knowledge of the relative equilibrium solutions is equivalent to the knowledge of the critical points of the normalized potential $\bar{V}$ (see [5], [3], [1]). The potential $V = -\sum_{i<j} m_i m_j / r_{ij}$, $r_{ij} = |x_i - x_j|$, where $x_i \in R^2$ is the position of m_i with respect to the center of mass, is normalized by keeping the moment of inertia $I = (1/2) \sum_{i<j} m_i m_j r_{ij}^2$ equal to 1.

It is well known that there are exactly five critical points: 2 Lagrangian points with the masses forming an equilateral triangle and 3 Eulerian points with masses on a line. The masses can be viewed as the barycentric coordinates of a point of a triangle T. We shall refer to that triangle as the mass triangle.

Let I_{hc} be the set of points of the phase space with energy h and angular momentum c. We call loosely I_{hc} an invariant manifold because it is invariant under the three-body flow. It is not true that I_{hc} is a manifold for all (h, c) values.

We hope that the knowledge of the topology of I_{hc} can help to fully understand the flow in the three-body problem. The study of I_{hc} for different (h, c) pairs requires to know the order of the normalized potentials. The first objective of this note is to prove the following theorem that was conjectured by Smale [5].

Indiana University Mathematics Journal ©, Vol. 30, No. 1 (1981)