ON TOPOLOGICAL ENTROPY, LEFSCHETZ NUMBERS AND LEFSCHETZ ZETA FUNCTIONS

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ABSTRACT. In the present article we give sufficient conditions for C^∞ self-maps on some connected compact manifold manifolds in order to have positive entropy. The conditions are given in terms of the Lefschetz numbers of the iterates of the map and/or its Lefschetz zeta function. We consider the cases where the manifold is a compact orientable and non-orientable surface, the n-dimensional torus, the product of n spheres of dimension ℓ and the product of spheres of different dimensions.

1. Introduction

The topological entropy of a topological dynamical system is a non-negative real number which measures the complexity of the system. Topological entropy was first introduced in 1965 by Adler, Konheim and McAndrew [1]. Later on Dinaburg [8] and Bowen [7] provided a distinct, weaker definition inspired in the Hausdorff dimension. This new definition clarified the meaning of the topological entropy. Thus for a system defined by the iteration of a function, the topological entropy is the exponential growth rate of the number of distinguishable orbits of the iterates.

Here we provide sufficient conditions in order that the topological entropy of a map in a compact manifold be positive in terms of the Lefschetz numbers of the iterates of the map or in function of the Lefschetz zeta function of the map.

This paper is organized as follows: In section 2 we introduce basic notions and definitions, as entropy, Lefschetz numbers and Lefschetz zeta functions, we also describe the entropy conjecture. Later in this section we prove Theorem 6 which gives sufficient conditions for a C^∞

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self-maps on a manifold to have positive topological entropy in terms of its Lefschetz zeta function. In section 3 we consider the case of compact and connected surfaces, either orientable and non-orientable, with and without boundary. We give sufficient conditions for C^0, as well as C^∞, self-maps on a surface in order to have positive entropy. The case of orientable surfaces are considered in Theorems 7 and 8 and non-orientable surfaces in Theorems 9 and 3. In Theorems 7 and 9 the conditions are given in terms of the Lefschetz zeta functions of the map, these two theorems are based on Theorem 6 and in Theorems 8 and 3 the conditions are given in terms of their Lefschetz numbers.

In section 4 we consider the case of the n-dimensional torus, since the entropy conjecture is true for continuous self-maps on $\mathbb{T}^n$, we consider in this section continuous maps on this manifold. In Theorem 11 we give the sufficient conditions for a continuous map to have positive entropy in terms of their Lefschetz numbers.

In section 5 we consider the case of the product of spheres. First we study the product of spheres of the same dimension. In Theorem 14 a criteria is given in terms of the Lefschetz numbers of the map. We also consider the case of product of spheres of different dimensions, which is more elaborated from the combinatorial point of view. In Theorem 16 the conditions are given in terms of the Lefschetz numbers, where as Corollary 17 deals with the Lefschetz zeta functions of the maps. All maps considered in this section are C^∞.

In section 6, we give some remarks about the relationship between the entropy of a map and the factorization of its Lefschetz zeta function in factors of the form $(1 \pm t^{d_i})^{\pm 1}$, with positive integers d_i.

Some results related with the ones of this paper can be found in [12, 15, 17, 19, 24].

2. Basic Notions, definitions and general results

Let X be a topological space and f a continuous self-map on X.

We give the Bowen's definition of the topological entropy (*cf.* [7]). We suppose that X is a compact metric space, with a metric d. We define the metric d_m on X as

$$d_m(x,y) := \max_{0 \le j \le m} d(f^j(x), f^j(y)).$$

A finite set S is called (m,ϵ)-*separated with respect to* f if for any $x, y \in S$ we have $d_m(x,y) > \epsilon$. We denote by $S(m,\epsilon)$ the maximal cardinality of an (m, ϵ)–separated set. The *topological entropy* of f is defined by

$$h(f) := \lim_{\epsilon \to 0} \left(\limsup_{m \to \infty} \frac{1}{m} \log S(m, \epsilon) \right).$$

We suppose that X is n–dimensional. The map f induces a homomorphism on the k–th rational homology group of X for $0 \le k \le n$, i.e. $f_{*k} : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})$. The $H_k(X, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$ and f_{*k} is a linear map whose matrix has integer entries.

The *spectral radii* of f_{*k} are defined as

$$\mathrm{sp}(f_{*k}) := \max \{ |\lambda| \, : \, \lambda \text{ is an eigenvalue of } f_{*k} \},$$

and we call the *spectral radius* of f_* (sometimes it s also called the spectral radius of f):

$$\mathrm{sp}(f_*) := \max_{0 \le k \le n} \mathrm{sp}(f_{*k}).$$

There is a famous and long standing conjecture called the *entropy conjecture* that states that if f is a C^1 self-map on a compact manifold then

$$\log(\mathrm{sp}(f_*)) \le h(f).$$

This conjecture was stated by Shub in 1974 (*cf.* [23]). Yomdin proved it for C^∞ maps (*cf.* [25]):

Theorem 1 (Yomdin)**.** *If f is a C^∞ self maps on a compact manifold, then $h(f) \ge \log(\mathrm{sp}(f_*))$.*

In the case of self-maps on the n-dimensional torus the conjecture was proved for continuous maps by Misiurewicz and Przytycki (*cf.* [22]). The conjecture is also true for continuous self-maps on nilmanifolds (*cf.* [21]).

Other important result related to this matter is the following theorem, due to Manning (*cf.* [20]):

Theorem 2 (Manning)**.** *Let f be a C^0 self-map on a compact manifold then $h(f) \ge \log(\mathrm{sp}(f_{*1}))$.*

The *Lefschetz number* of f is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}(f_{*k}).$$

The Lefschetz Fixed Point Theorem states that if $L(f) \neq 0$ then f has a fixed point (*cf.* [6] or [16]).

The *Lefschetz zeta function* of f is defined as

$$\zeta_f(t) = \exp\left(\sum_{m\geq 1} \frac{L(f^m)}{m}\, t^m\right).$$

Since $\zeta_f(t)$ is the generating function of the Lefschetz numbers, $L(f^m)$, it keeps the information of the Lefschetz number for all the iterates of f. There is an alternative way to compute it:

$$\zeta_f(t) = \prod_{k=0}^{n} \det(Id_k - tf_{*k})^{(-1)^{k+1}}, \tag{1}$$

where $n = \dim X$, $m_k = \dim H_k(X, \mathbb{Q})$, Id_k is the identity map on $H_k(X, \mathbb{Q})$, and by convention $\det(Id_k - tf_{*k}) = 1$ if $m_k = 0$ (*cf.* [10]).

We mention an elementary fact about the Lefschetz zeta function that we use in this article.

Proposition 3. *Let $f : X \to X$ be a continuous map. If $\lambda \neq 0$ is a zero or pole of $\zeta_f(t)$ then λ^{-1} is an eigenvalue of f_{*k}, for some k. Moreover 0 is never a zero or pole of $\zeta_f(t)$. In particular if the eigenvalues of f_* are roots of unity then the zeroes and poles of $\zeta_f(t)$ are also roots of unity.*

Proof. Let $q_k(t) = \det(Id_k - tf_{*k})$. Note that $q_k(0) = 1$, for all k; so from (1), we get $\zeta_f(0) = 1$.

If $\lambda \neq 0$ is a zero or pole of $\zeta_f(t)$ then $q_k(\lambda) = 0$ for some k. Hence $\det(\lambda^{-1} Id_{*k} - f_{*k}) = 0$, i.e. λ^{-1} is an eigenvalue of f_{*k}.

Note that ω is a root of unity if and only if ω^{-1} is a root of unity. Hence if all eigenvalues of f_* are root of unity then the zeroes or poles are also roots of unity. □

From (1) it follows the general expression of a continuous self-map on X is either $\zeta_f(t) = 1$ or of the form

$$\zeta_f(t) = \prod_{i=1}^{N} q_i(t)^{s_i},$$

where $q_i(t) = c_{i,m_i} t^{m_i} + \cdots + c_{i,0}$ is a polynomial not necessary monic, however $c_{i,0} = 1$, for all $1 \leq i \leq N$, we will always assume that the polynomials $q_i(t)$ do not have common factors, and the numbers s_i are integers.

We remark that if $p_i(t)$ is the characteristic polynomial of f_{*i}, its degree is $n_i = \dim(H_i(X, \mathbb{Q}))$, and $q_i(t) = \det(Id_i - tf_{*i})$. Then

$q_i(1/t) = t^{-n_i}p(t)$, if not all the eigenvalues of f_{*i} are zero. Note that $q_i(t)$ is a polynomial of degree at most n_i, and it is equal to n_i if and only if zero is not a root of $p_i(t)$.

First we show two propositions that relate the sizes of the coefficients of the polynomials with their roots. We shall use these facts throughout the article.

Proposition 4. *Let $s(t) = t^m + \alpha_{m-1}t^{m-1} + \cdots + \alpha_1 t + \alpha_0$ be a monic polynomial with integers coefficients. Suppose that all its roots are zero or roots of unity.*

(a) *If $\alpha_0 \neq 0$ then $|\alpha_{m-k}| \leq \binom{m}{k}$ for $1 \leq k \leq m-1$.*
(b) *If $\alpha_0 = \alpha_1 = \cdots = \alpha_{r-1} = 0$ and $\alpha_r \neq 0$ then $|\alpha_{m-k}| \leq \binom{m-r}{k}$ for $1 \leq k \leq m-r$.*

Proof. Vieta's formulae allow to express the coefficient of the polynomial $s(t)$ in term of its roots. Let $\lambda_1, \ldots, \lambda_m$ be the roots of the polynomial $s(t)$.

The Vieta's formulae are

$$\begin{aligned} \alpha_0 &= \lambda_1 \cdots \lambda_m, \\ \alpha_1 &= \sum_{i_1 < \cdots < i_{m-1}} \lambda_{i_1} \cdots \lambda_{i_{m-1}}, \\ &\vdots \\ \alpha_{m-2} &= \sum_{i<j} \lambda_i \lambda_j, \\ \alpha_{m-1} &= \lambda_1 + \cdots + \lambda_m. \end{aligned} \tag{2}$$

From identity (2) and using the fact that all the roots of $s(t)$ are roots of unity, we have

$$|\alpha_{m-1}| \leq \sum_{i=1}^{m} |\lambda_i| = m.$$

For the general term:

$$|\alpha_{m-k}| \leq \sum_{1 \leq i_1 < \cdots < i_k \leq m} |\lambda_{i_1} \cdots \lambda_{i_k}| = \sum_{1 \leq i_1 < \cdots < i_k \leq m} 1 = \binom{m}{k}.$$

This proves statement (a).

If $\alpha_0 = \alpha_1 = \cdots = \alpha_{r-1} = 0$ and $\alpha_r \neq 0$ then $s(t) = t^r u(t)$, where

$$u(t) = t^{m-r} + \alpha_{m-1-r}t^{m-1-r} + \cdots + \alpha_{r+1}t + \alpha_r.$$

Applying part (a), to the polynomial $u(t)$ we get statement (b). □

Proposition 5. *Let $s(t) = t^m + \alpha_{m-1}t^{m-1} + \cdots + \alpha_1 t + \alpha_0$ be a monic polynomial with integers coefficients, and $q(t) = c_{m-r}t^{m-r} + \cdots + c_0$ a polynomial with $c_0 = 1$, such that they satisfy $q(t) = t^r s(t^{-1})$.*

If $|c_j| > \binom{m-r}{j}$ for some $1 \leq j \leq m-r-1$. Then there exists λ root of $s(t)$ satisfying $|\lambda| > 1$ and moreover λ^{-1} is a root of $q(t)$.

Proof. If the polynomials $s(t)$ and $q(t)$ satisfy the relation $q(t) = t^r s(t^{-1})$ then $c_{m-j} = \alpha_j$ for $0 \leq j \leq r$.

From Proposition 4 it follows that if all the roots of $s(t)$ are zero or they have norm equal 1 then $|\alpha_{m-j}| \leq \binom{m-r}{j}$ for $1 \leq j \leq m-r$. Hence $|c_j| \leq \binom{m-r}{j}$. Therefore the condition $|c_j| > \binom{m-r}{j}$ implies that $s(t)$ has a root λ, with $|\lambda| \neq 1$. Since α_0 is an integer and from formula (2) we can conclude that $|\lambda| > 1$. Clearly λ^{-1} is a root of $q(t)$. $\square$

Theorem 6. *Let f be a C^∞ self-map on compact connected differentiable manifold X, whose Lefschetz zeta function is of the form*

$$\zeta_f(t) = \prod_{i=1}^{N} q_i(t)^{s_i}, \tag{3}$$

where N a positive integer, the s_i are integers and $q_i(t)$ are polynomials of the form:

$$q_i(t) = c_{i,m_i}t^{m_i} + \cdots + c_{i,0}, \tag{4}$$

where the coefficients $c_{i,j}$ are integers, and in particular $c_{i,0} = 1$.

If the polynomials $q_i(t)$ do not have common factors and for some (i,j), i.e. $q_i(t)$ and $q_k(t)$ are co-prime, for every $i \neq k$. there exists $|c_{i,j}| > \binom{m_i}{j}$, then the map f has positive topological entropy.

Proof. Suppose that in the expression of the Lefschetz zeta function of f as in (3), there is a polynomial $q_i(t)$ as in (4), with $c_{i,0} = 1$. Hence

$$q_i(t) = c_{i,m_i}t^{m_i} + \cdots + c_{i,0}.$$

By Proposition 5, if $|c_{i,j}| > \binom{m_i}{j}$ then there exists λ a root of $q_i(t)$, such that $0 < |\lambda| < 1$. Hence λ^{-1} is an eigenvalue of f_{*k}, for some k. From Theorem 1, we can conclude that the map f has positive entropy. $\square$

3. Surfaces

Let $X = X^+(g,b)$ be a compact connected orientable surface of genus $g \geq 0$ with $b \geq 0$ boundary components, its homology groups are:

$$H_k(X^+(g,b),\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k=0, \\ \overbrace{\mathbb{Q}\oplus\cdots\oplus\mathbb{Q}}^{2g-\text{times}} & \text{if } k=1 \text{ and } b=0, \\ \overbrace{\mathbb{Q}\oplus\cdots\oplus\mathbb{Q}}^{(2g+b-1)-\text{times}} & \text{if } k=1 \text{ and } b>0, \\ \mathbb{Q} & \text{if } k=2 \text{ and } b=0, \\ 0 & \text{if } k=2 \text{ and } b>0. \end{cases}$$

Let f be a continuous self-map on $X^+(g,b)$. If $b=0$ its Lefschetz zeta function is

$$\zeta_f(t) = \frac{q(t)}{(1-t)(1-Dt)},$$

where D is the degree of f, i.e. the map $f_{*2} = (D)$ and $q(t) = \det(Id - f_{*1}t)$. Let $p(t)$ be the characteristic polynomial of f_{*1}, its degree is $2g$, then $q(1/t) = t^{-2g}p(t)$, if not all the eigenvalues of f_{*1} are zero. Note that $q(t)$ is a polynomial of degree at most $2g$, and it is equal to $2g$ if and only if zero is not a root of $p(t)$. In the case $b>0$ the Lefschetz zeta function is

$$\zeta_f(t) = \frac{q(t)}{1-t},$$

where $q(t) = \det(Id - f_{*1}t)$, it is a polynomial of degree at most $2g + b - 1$.

As it can be seen in the previous remark, the eigenvalues of the induced map on the first homology play a very important role in the expression of the Lefschetz zeta function. Therefore Theorem 2 is very relevant in this situation. For this reason we can consider C^0 maps. The adapted version of Theorem 6 for self-maps on $X^+(g,b)$ is the following statement:

Theorem 7. *Let f be a C^0-self map on $X^+(g,b)$.*

(a) *Let $b=0$ and the Lefschetz zeta function of f:*

$$\zeta_f(t) = \frac{q(t)}{(1-t)(1-Dt)},$$

where D is the degree of f and the polynomial $q(t)$ is of the form:

$$q(t) = c_{m-r}t^{m-r} + \cdots + c_0,$$

with $c_0 = 1$, $m \leq 2g$ and $0 \leq r \leq m-1$. If $|c_j| > \binom{m-r}{j}$ for some j such that $1 \leq j \leq m-r-1$, then the map f has positive topological entropy.
Moreover, if f is C^∞ and $D \neq 0, 1, -1$ then the topological entropy of f is positive.

(b) *Let $b > 0$ and the Lefschetz zeta function of f:*

$$\zeta_f(t) = \frac{q(t)}{1-t},$$

where the polynomial $q(t)$ is of the form:

$$q(t) = c_{m-r}t^{m-r} + \cdots + c_0,$$

with $c_0 = 1$, $m \leq 2g+b-1$ and $0 \leq r \leq m-1$. If $|c_j| > \binom{m-r}{j}$ for some $1 \leq j \leq m-r-1$, then the map f has positive topological entropy.

Proof. Since the polynomial $q(t)$ appears in the Lefschetz zeta funtion, is $q(t) = \det(Id - tf_{*1})$. Hence $p(t)$ the characteristic polynomial of f_{*1} is of the form

$$p(t) = t^r(t^{m-r} + a_{m-1}t^{m-r-1} + \cdots + a_r),$$

where $c_j = a_{m-j}$ for $1 \leq j \leq m-r$.

We apply Proposition 4 to the polynomial $s(t) = t^{m-r} + a_{m-1}t^{m-r-1} + \cdots + a_r$. Hence if $r = 0$, i.e. $a_0 \neq 0$, and all the roots of $s(t)$ are roots of unity then $|a_{m-1}| \leq m$. So the condition $|a_{m-1}| > m$, which is $|c_1| > m$, implies that there exits at least one eigenvalue of f_{*1} of modulus greater than 1; hence, by Theorem 2 we conclude $h(f) > 0$. Similarly if $|a_{m-j}| > \binom{m}{j}$, i.e. $|c_j| > \binom{m}{j}$, implies $h(f) > 0$.

If $r > 0$ then $a_0 = a_1 = \cdots = a_{r-1} = 0$. By Proposition 4 $|a_m| \leq m-r$ if all the roots of $s(t)$ are roots of unity. Hence the condition $|a_m| > m-r$ implies that there exits an eigenvalue of f_{*1} outside the unit circle, by Theorem 2 the map f has positive topological entropy. Similarly $|a_{m-j}| > \binom{m-r}{j}$, i.e. $|c_j| > \binom{m-r}{j}$, implies $h(f) > 0$.

On the other hand, since $f_{*2} = (D)$ the condition $D \notin \{0, 1, -1\}$, implies that the spectral ratio of f_* is greater than 1. If f is C^∞ then $h(f) > 0$, due to Theorem 1. This proves statement (a).

Statement (b) is proved in a similar manner as in statement (a), therefore we omit its proof. □

We can also express a sufficient condition for having positive entropy in terms of the Lefschetz numbers.

Theorem 8. *Let f be a C^∞ self-map on $X^+(g,b)$. If $b=0$ and $|L(f)|>2(g+1)$ then f has positive entropy.*

Moreover, if f is C^0 and $|L(f)| > 2g+1+|D|$ then f has positive entropy.

If f is a C^0 self-map on $X^+(g,b)$ with $b>0$ and $|L(f)|>2g+b$ then f has positive topological entropy.

Proof. From the definition of the Lefschetz number, we have

$$\begin{aligned} L(f) &= \operatorname{trace}(f_{*0}) - \operatorname{trace}(f_{*1}) + \operatorname{trace}(f_{*2}) \\ &= 1-(\lambda_1+\cdots+\lambda_m)+D. \end{aligned}$$

where $m=\dim(H_1(X^+(g,b)))$, i.e. $m=2g$ if $b=0$ and $m=2g+b-1$ if $b>0$, and $D=0$ if $b>0$.

If all the eigenvalues of f_{*1} are root of unity or zero then

$$|L(f)| \le 1+|\lambda_1|+\cdots+|\lambda_m|+|D| \le 1+m+|D|.$$

Hence in the case $b=0$, if $|L(f)|>1+2g+|D|$ then the spectral radius of f_{*1} is greater than 1. From Theorem 2 it follows f has positive topological entropy. However if f is a C^∞ map and $|D|>1$ then $h(f)>0$; therefore the condition $|L(f)|>2(1+g)$ implies that one of the eigenvalues of f_* is outside of the unit circle, hence $h(f)>0$.

In the case $b>0$, we have $D=0$, so the condition $|L(f)|>2g+b$ implies the existence of an eigenvalue of f_{*1} of modulus greater then 1, hence it has positive topological entropy. □

Let $X=X^-(g,b)$ be a compact, connected, non–orientable surface of genus $g\ge 1$ with $b\ge 0$ boundary components, its homology groups are:

$$H_k(X^-(g),\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k=0; \\ \overbrace{\mathbb{Q}\oplus\cdots\oplus\mathbb{Q}}^{(g+b-1)-\text{times}} & \text{if } k=1. \\ 0 & \text{if } k=2. \end{cases}$$

Let f be a continuous self-map on $X^-(g,b)$, its Lefschetz zeta function is given by:

$$\zeta_f(t) = \frac{q(t)}{1-t},$$

where $q(t) = \det(Id - f_{*1}t)$. Let $p(t)$ be the characteristic polynomial of f_{*1}, its degree is $g+b-1$, then $q(1/t) = t^{-(g+b-1)}p(t)$. If not all the eigenvalues of f_{*1} are zero. As in the orientable case $q(t)$ is a polynomial of degree at most $g+b-1$, and it is equal to $g+b-1$ if and only if zero is not a root of $p(t)$.

Theorem 9. *Let f be a C^0 self-map on $X^-(g,b)$ with Lefschetz zeta function*

$$\zeta_f(t) = \frac{q(t)}{1-t},$$

where $q(t)$ is a polynomial of the form:

$$q(t) = c_{m-r}t^{m-r} + \cdots + c_0,$$

with $c_0 = 1$, $m \le g+b-1$ and $0 \le r \le m-1$. If $|c_j| > \binom{m-r}{j}$ for some j such that $1 \le j \le m-r-1$, then the map f has positive topological entropy.

The proof of this theorem follows the same arguments as the proof of Theorem 7, for this reason we omit it here.

Theorem 10. *Let f be a C^0 self-map of $X^-(g,b)$. If $|L(f)| > g+b$ then f has positive entropy.*

Proof. From the definition of the Lefschetz number, we have

$$\begin{aligned} L(f) &= \operatorname{trace}(f_{*0}) - \operatorname{trace}(f_{*1}) + \operatorname{trace}(f_{*2}) \\ &= 1 - (\lambda_1 + \cdots + \lambda_{g+b-1}) + 0, \end{aligned}$$

where the λ_i's are the eigenvalues of f_{*1}.

If $|L(f)| > g+b$ then there exits one λ_i with $|\lambda_i| > 1$. Hence the topological entropy of f is positive. □

4. The n-dimensional torus: $\mathbb{T}^n$

In this section we set $X = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n-\text{times}} = \mathbb{T}^n$, the n dimensional torus.

As it was mention in section 2, in the entropy conjecture is true for continuous self-maps on $\mathbb{T}^n$ (*cf.* [22]). Therefore, in this section we consider continuous maps, i.e. C^0 maps.

Theorem 11. *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a continuous map. If $|L(f)| > 2^n$ then $h(f) > 0$.*

Proof. The Lefschetz numbers for f satisfy $L(f) = \prod_{i=1}^{n}(1-\lambda_i)$, where the λ_i's are the eigenvalues of f_{*1} (*cf.* [5]). If all the eigenvalues of f are zero or root of unity then

$$|L(f)| \leq \prod_{i=1}^{n} |1+\lambda_i| \leq \prod_{i=1}^{n}(1+|\lambda_i|) \leq 2^n.$$

Hence if $|L(f)| > 2^n$ there exits $i \in \{1,\dots,n\}$ such that $|\lambda_i| > 1$. Therefore, from Theorem 1, it follows the topological entropy of f is positive. □

Now we will consider criteria for a continuous self-map on $\mathbb{T}^n$ to have positive entropy in terms of the Lefschetz zeta function. It is known that the Lefschetz zeta function for a map whose eigenvalues are not zero, is a rational function such that the degree of the polynomial on the numerator and the degree of the polynomial on the denominator are the same (*cf.* [2]). Furthermore if the eigenvalues of f_{*1} are root of unity then the Lefschetz zeta function of f is of the form (*cf.* [3]):

$$\zeta_f(t) = \prod_{i=1}^{N}(1 \pm t^{d_i})^{r_i},$$

or $\zeta_f(t) = 1$, with positive integers d_i and integers r_i. Moreover in [3] there are explicit formulae for the values of d_i and r_i in terms of the characteristic polynomial of f_{*1}.

Proposition 12. *Let f be a continuous self-map on $\mathbb{T}^n$ if its Lefschetz zeta function is not constant and it cannot be factorized in the form*

$$\zeta_f(t) = \prod_{i=1}^{N}(1 \pm t^{d_i})^{r_i}, \tag{5}$$

where N, d_i are positive integers and r_i integers. Then $h(f) > 0$.

Proof. If $h(f) = 0$ then all the eigenvalues of f_{*1} are roots of unity. It is well known that 1 is an eigenvalue of f_{*1} if and only if $\zeta_f(t) = 1$ (*cf.* [13]). So if 1 is not an eigenvalue of f_{*1} then the Lefschetz zeta function of f can be factorized in the form of

$$\zeta_f(t) = \prod_{i=1}^{N}(1 \pm t^{d_i})^{r_i},$$

with N, d_i positive integers and r_i integers, ([3]). □

Theorem 6 is valid for continuous self maps on the n-dimensional torus.

In the following example we show the importance of the fact that the polynomials $q_i(t)$ in the Lefschetz zeta function in the expression (3), do not have common factors. Let f be a map on $\mathbb{T}^3$ given by the integer matrix in $\mathbb{R}^3$:

$$A = \begin{pmatrix} R & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $R > 1$ is a integer.

The map f has positive entropy since the eigenvalues of f_{*1} are $R, 1$ and -1. However its Lefschetz zeta function is: $\zeta_f(t) = 1$, as many maps of topological entropy zero, for example if $R = \pm 1$. As we remarked before the Lefschetz zeta function, of a continuous self-map on $\mathbb{T}^n$, is equal 1 if and only if 1 is an eigenvalue of f_{*1} (*cf.* [13]).

5. The product of spheres

Let $X = \mathbb{S}^{\ell_1} \times \cdots \times \mathbb{S}^{\ell_n}$. Using the Künneth Theorem (see for instance [14]), we compute the homology groups of X over the rational numbers $\mathbb{Q}$. They are given by

$$H_k(X, \mathbb{Q}) = \begin{cases} \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{b_k} & \text{if } b_k \neq 0, \\ \{0\} & \text{if } b_k = 0, \end{cases}$$

with $0 \leq k \leq \ell_1 + \cdots + \ell_n$; where b_k is the number of ways that k can be obtained as by summing up subsets of $(\ell_1, \ldots, \ell_n)$, i.e. b_k is the cardinality of the set

$$\left\{ S \subset \{1, \ldots, n\} \, : \, \sum_{j \in S} \ell_j = k \right\}. \tag{6}$$

The numbers b_k are the *Betti numbers of* X.

First we consider the product of spheres of the same dimension: Let $X = X(\ell, n) = \underbrace{\mathbb{S}^\ell \times \cdots \times \mathbb{S}^\ell}_{n \text{ times}}$. If $\ell_1 = \cdots = \ell_n = \ell$, then

$$b_k = \begin{cases} \binom{n}{j} \text{ if } k = j\ell, \\ 0 \text{ if } k \text{ is not a multiple of } \ell. \end{cases}$$

For the Lefschetz numbers of self-maps on $X(\ell, n)$, we recall the following theorem:

Theorem 13 ([4]). *Let $f : X(\ell, n) \to X(\ell, n)$ be a continuous map. Then, for all $m > 0$,*

$$L(f^m) = \det(Id + (-1)^\ell f_{*\ell}^m),$$

where Id is the identity map on $\mathbb{Q}^n$. In particular,

$$\begin{cases} L(f^m) = C_{f_{*\ell}^m}(1) & \text{if } \ell \text{ is odd,} \\ L(f^m) = (-1)^n C_{f_{*\ell}^m}(-1) & \text{if } \ell \text{ is even,} \end{cases}$$

where $C_H(t) := \det(t\, Id - H)$ denotes the characteristic polynomial of the linear map H.

Theorem 14. *Let $f : X(\ell, n) \to X(\ell, n)$ be a C^∞ map. If $|L(f)| > 2^n$ then $h(f) > 0$.*

Proof. The proof goes along the same idea as in Theorem 11. By Theorem 13,

$$|L(f)| = |\det(\mathrm{Id} + (-1)^\ell f_{*\ell})| = |\prod_{i=1}^{n}(1 + (-1)^\ell \lambda_i)| \leq \prod_{i=1}^{n}(1 + |\lambda|^i|),$$

where the λ_i's are the eigenvalues of $f_{*\ell}$ So if $L(f) > 2^n$ there exists λ_i, for some $1 \leq i \leq n$, such that $|\lambda_i| > 1$. Hence $h(f) > 0$. □

As in the case of the n-dimensional torus if the eigenvalues of $f_{*\ell}$ are root of unity then the Lefschetz zeta function of f is of the form (*cf.* [4]):

$$\zeta_f(t) = \prod_{i=1}^{n}(1 \pm t^{d_i})^{r_i},$$

or $\zeta_f(t) = 1$, with d_i positive integers and r_i integers. So we have the following result, which is similar to Proposition 12.

Proposition 15. *Let f be a C^∞ self-map on $X(\ell, n)$ if its Lefschetz zeta function is not constant and it cannot be factorized in the form*

$$\zeta_f(t) = \prod_{i=1}^{n}(1 \pm t^{d_i})^{r_i},$$

with d_i are positive integers and r_i integers. Then $h(f) > 0$.

In the following lines we will consider the product of spheres with different dimensions, i.e. $X = \mathbb{S}^{\ell_1} \times \cdots \times \mathbb{S}^{\ell_n}$, with $\ell_1 < \cdots < \ell_n$.

Let

$$M = M(\ell_1, \dots, \ell_n) := \bigcup_{s=1}^{n} \{\ell_{i_1} + \cdots + \ell_{i_s} \, : \, i_1 < \cdots < i_s\}.$$

We suppose that $\ell_1 < \cdots < \ell_n$. By elementary combinatorics we have that the cardinality of the set M is at most $2^n - 1$. We shall assume that the cardinality of M is exactly $2^n - 1$, i.e. the numbers $\ell_1, \dots, \ell_n$ are such that all the sums defined in the set M are different. In this condition is satisfied we say that M is a *sum free set*. In this case the Betti numbers of X are $b_k = 1$ if $k \in M$ and $b_k = 0$ otherwise, i.e.

$$H_k(X, \mathbb{Q}) = \mathbb{Q} \text{ if } k \in \{0\} \bigcup_{s=1}^{n} \{\ell_{i_1} + \cdots + \ell_{i_s} \,|\, i_1 < \cdots < i_s\},$$

and trivial otherwise.

Let $f : X \to X$ be a continuous map, since the homology groups are either 1-dimensional or trivial, its induced maps on homology are $f_{*k} = (a_k)$ if $k \in M$ and $f_{*k} = 0$ otherwise.

Theorem 16. *Let $X = \mathbb{S}^{\ell_1} \times \cdots \times \mathbb{S}^{\ell_n}$ and f be a C^∞ self-map on X. We assume that the set of partial sums of the indexes of the dimension of the spheres, $M(\ell_1, \dots, \ell_n)$, is a sum free set. If $L(f) > 2^n$ then the topological entropy of f is positive.*

Proof. The Lefschetz numbers for f are:

$$L(f^m) = 1 + \sum_{k \in M} (-1)^k a_k^m,$$

for all integer $m > 0$. So $|L(f)| \leq 1 + \sum_{k \in M} |a_k|$. Since the cardinality of the set M is $2^n - 1$, if the condition $|L(f)| > 2^n$ holds, then there exist $j \in M$, such that $|a_j| > 1$. Then the positive entropy of f is positive. □

According to (1) the Lefschetz zeta function of the map f in the case of the sum-free condition is

$$\zeta_f = (1 - a_0 t)^{-1} \prod_{k \in M} (1 - t a_k)^{(-1)^{k+1}} = \frac{\displaystyle\prod_{k \in M} (1 - t a_k)^{(-1)^{k+1}}}{1 - t}.$$

Clearly if a_k are different from $0, 1$ or -1 then f has positive entropy.

In the following lines we describe the Lefschetz zeta function of f, when the sum-free condition is not satisfied. For this we need to give a partition of the set M. We recall that the Betti number b_k is the cardinality of the set described in (6). If $r \neq 0$, we introduce

$$M_r := \left\{ k = \sum_{j \in S} \ell_j \;:\; S \subset \{1, \dots, n\},\, |S| = r \right\}.$$

In other words the set M_r consists of the numbers k's such that can be obtained as r different sums of elements of the set $\{\ell_1, \dots, \ell_n\}$. Clearly they form a disjoint partition of M: $M = \cup_{r=1}^{R} M_r$, where $R = \max_{1 \le k \le L} b_k$ and $L = \ell_1 + \cdots + \ell_n$. The set M is sum-free if and only if $M = M_1$. From the definition follows that $k \in M_r$ if and only if $b_k = r$.

We will give an example: Let $\ell_1 = 1, \ell_2 = 3, \ell_3 = 5, \ell_4 = 6$, them

$$M_1 = \{1, 3, 4, 5, 7, 8, 9, 10, 11, 14, 15\}, \quad M_2 = \{6, 9\};$$

since $6 = \ell_4 = \ell_1 + \ell_3$ and $9 = \ell_2 + \ell_4 = \ell_1 + \ell_2 + \ell_3$.

From (1) the Lefschetz zeta function of the map f, $\zeta_f(t)$ is of the form:

$$\prod_{k=0}^{L} \det(Id_k - tf_{*k})^{(-1)^{k+1}} = (1-t)^{-1} \prod_{k \in M} \det(Id_k - tf_{*k})^{(-1)^{k+1}} =$$

$$= (1-t)^{-1} \prod_{r=1}^{R} \prod_{k \in M_r} \det(Id_k - tf_{*k})^{(-1)^{k+1}} =$$

$$\frac{\prod_{k \in M_1}(1 - ta_k) \prod_{k \in M_2} \det(Id_k - tf_{*k})^{(-1)^{k+1}} \cdots \prod_{k \in M_R} \det(Id_k - tf_{*k})^{(-1)^{k+1}}}{1-t}.$$

Let $q_k(t) := \det(Id_k - tf_{*k})$. Note that $\det(Id_k - tf_{*k})$ is a polynomial of degree at most b_k if $k \in M_{b_k}$. So

$$q_k(t) = c_{k, m_k - r_k} t^{m_k - r_k} + \cdots + c_{k,0},$$

with $c_{k,0} = 1$ and $m_k - r_k \le b_k$, if $k \in M_{b_k}$.

In the example of $X = \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{S}^5 \times \mathbb{S}^6$, here $L = 15$ and $R = 2$, since $b_k = 1$ for $k \in M_1$, $b_k = 2$ for $k \in M_2$, and $b_k = 0$ otherwise. The Lefschetz zeta function of a self map f, $\zeta_f(t)$, on this space is of the form:

$$\prod_{k=0}^{15} \det(Id_k - tf_{*k})^{(-1)^{k+1}} =$$

$$(1-t)^{-1} \prod_{k \in M_1} \det(Id_k - tf_{*k})^{(-1)^{k+1}} \prod_{k \in M_2} \det(Id_k - tf_{*k})^{(-1)^{k+1}} =$$

$$\frac{(1-ta_1)(1-ta_3)(1-ta_5)(1-ta_7)(1-ta_9)(1-ta_{11})(1-ta_{15})}{(1-t)(1-ta_4)(1-ta_8)(1-ta_{10})(1-ta_{14})} \cdot \frac{q_9(t)}{q_6(t)},$$

where

$$q_6(t) = c_{6,2}t^2 + c_{6,1}t + c_{6,0}, \quad q_9(t) = c_{9,2}t^2 + c_{9,1}t + c_{9,1},$$

with $c_{6,0} = c_{9,0} = 1$.

Theorem 6 for self-maps on $X = \mathbb{S}^{\ell_1} \times \cdots \times \mathbb{S}^{\ell_n}$ is the following statement:

Corollary 17. *Let f be a C^∞ self-map on $X = \mathbb{S}^{\ell_1} \times \cdots \times \mathbb{S}^{\ell_n}$.*

(a) *If $|a_k| > 1$, for some $k \in M_1$ then $h(f) > 0$.*
(b) *If the polynomials $q_k(t)$ do not have common roots and there exits a coefficient $c_{k,j}$, with $|c_{k,j}| > \binom{m_k - r_k}{j}$. Then the topological entropy of f is positive.*

6. Remarks

For self-maps on $\mathbb{T}^n$ and $X(\ell, n)$, we have seen in Propositions 12 and 15, that if its Lefschetz zeta functions are not constant and they cannot be factorized on the form (5), i.e.

$$\zeta_f(t) = \prod_{i=1}^{N} (1 \pm t^{d_i})^{r_i},$$

with positive integers N, d_i and integers r_i; then the maps have positive entropy. In general if the Lefschetz zeta function is factorized as in (5), i.e. all its zeros and poles are roots of unity, it does not mean that the spectral radius of f_* is greater than 1. Since there could be eigenvalues λ_i of f_{*k}, for some k, such that the corresponding factors $1 - \lambda_i t$ in $\zeta_f(t)$ cancel out. The maps whose Lefschetz zeta function is of the form (5) or constant, are called *almost quasi-unipotent*, and they were studied in [18]. The map shown in the example given in section 4 is an almost quasi-unipotent map, with an eigenvalue of modulus bigger than 1.

We would like to remark the fact that the factorization of the Lefschetz zeta function in the form (5) is a necessary condition for a map to have a finite number of periodic points (*cf.* [9, 11]), provided that the space is a differentiable manifold and the map differentiable.

As for necessary conditions for a map to have positive entropy, we wonder if the reciprocal of Proposition 12 is true.

Question: Let f be a continuous self-map on $\mathbb{T}^n$ with positive topological entropy and 1 is not an eigenvalue of f_{*1}. Does $\zeta_f(t)$ have a pole or zero which is not root of unity?

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References

[1] R.L. Adler, A.G. Konheim and M.H. McAndrew, Topological Entropy. Trans. Amer. Math. Soc. **114** (1965), 309–319.

[2] P. Berrizbeitia and V.F Sirvent, On the Lefschetz zeta function for quasi-unipotent maps on the n-dimensional torus. *J. Difference Equ. Appl.* **20** (2014), no. 7, 961–972.

[3] P. Berrizbeitia, M.J. González, A. Mendoza and V.F. Sirvent, On the Lefschetz zeta function for quasi-unipotent maps on the n-dimensional torus. II: The general case, *Topology Appl.* **210** (2016), 246–262.

[4] P. Berrizbeitia, M.J. González and V.F. Sirvent, On the Lefschetz zeta function and the minimal sets of Lefschetz periods for Morse-Smale diffeomorphisms on products of ℓ-spheres, *Topology Appl.* **235** (2018), 428–444.

[5] R.B.S. Brooks, R.F. Brown, J. Pak and D.H. Taylor, Nielsen numbers of maps of tori, *Proc. Am. Math. Soc.* **52** (1975) 398–400.

[6] R.F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman and Company, Glenview, IL, 1971.

[7] R. Bowen, Entropy for group endomorphisms snd homogenous spaces, *Trans. Amer. Math. Soc.* **153** (1971), 401–414.

[8] E. Dinaburg, Relationship between topological entropy and metric entropy, Doklady Akademii Nauk SSSR. **170** (1970), 19.

[9] J. M. Franks, Some smooth maps with infinitely many hyperbolic periodic points, *Trans. Am. Math. Soc.* **226** (1977), 175–179.

[10] J. Franks, *Homology and dynamical systems*, CBSM Regional Conf. Ser. in Math. **49**, Amer. Math. Soc., Providence, R.I. 1982.

[11] D. Fried, *Periodic points and twisted coefficients*, Lecture Notes in Maths., no 1007, Springer Verlag, 1983, 175–179.

[12] J.L. García G. and J. Llibre Topological entropy and periods of self-maps on compact manifolds. To appear in *Houston J. Math.*

[13] B. Halpern, Periodic points on tori, *Pac. J. Math.* **83** (1979), 117–133.

[14] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[15] J. Jezierski and W. Marzantowicz, *Homotopy methods in topological fixed and periodic points theory*, Springer Verlag, Berlin, 2006.

[16] S. Lefschetz, Intersections and transformations of complexes and manifolds, *Trans. Amer. Math. Soc.* **28** (1926), 1–49.

[17] J. Llibre, Brief survey on the topological entropy, *Discrete Contin. Dyn. Syst. Ser. B* **20** (2015), 3363–3374.
[18] J. Llibre and V.F. Sirvent, C^1 self-maps on closed manifolds with finitely many periodic points all of them hyperbolic. *Math. Bohem.* **141** (2016), 83–90.
[19] J. Llibre and V.F. Sirvent, On Lefschetz periodic point free self-maps,*J. Fixed Point Theory Appl.* **20** (2018), Art. 38, 9 pp.
[20] A. Manning, Topological entropy and the first homology group, in *Dynamical systems – Warwick 1974*, Lecture Notes in Math. 468, Springer–Verlag, Berlin, 1975, 185–190.
[21] W. Marzantowicz and F. Przytycki, Entropy conjecture for continuous maps on nilmanifolds, *Israel J. Math.* **165** (2008), 349–379.
[22] M. Misiurewicz and F. Przytycki, Entropy conjecture for tori. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **25** (1977), 575–578.
[23] M. Shub, Dynamical systems, filtrations and entropy, *Bull. Amer. Math. Soc.*, **80** (1974), 27–41.
[24] M. Shub and D. Sullivan, Homology theory and dynamical systems, *Topology* **14** (1975), 109–132.
[25] Y. Yomdin, Volume growth and entropy, *Israel J. Math.* **57** (1987), 285–300.

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