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On the stable limit cycle of a weight-driven pendulum clock

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Abstract

In a recent paper (Denny 2002 *Eur. J. Phys.* **23** 449–58), entitled 'The pendulum clock: a venerable dynamical system', Denny showed that in a first approximation the steady-state motion of a weight-driven pendulum clock is shown to be a stable limit cycle. He placed the problem in a historical context and obtained an approximate solution using the Green function. In this paper we obtain the same result with an alternative proof via known issues of classical averaging theory. This theory provides a useful means to study a planar differential equation derived from the pendulum clock, accessible to Master and PhD students.

1. Introduction and statement of the results

For students, weight-driven pendulum clocks provide an interesting, practical and historically important dynamical system to be considered. In a nice paper, Denny [3] showed that in a first approximation the steady-state motion of a weight-driven pendulum clock is shown to be a stable limit cycle. He obtains an approximate solution using the Green function. Here we obtain the same result with an alternative proof via the averaging theory.

The linearized equation of the pendulum with friction and escapement is

$$\ddot{\theta} + b\dot{\theta} + \omega^2\theta \approx \frac{1}{\Delta t} p(t, \dot{\theta}), \tag{1}$$

where b is the friction coefficient and the right-hand side of this expression is the escapement. We expect small pendulum amplitudes for grandfather clocks, <5, and so the linear approximation is a very good one. The function $p(t, \dot{\theta})$ represents the (angular) momentum transferred to the pendulum by the escapement mechanism, during the short time interval Δt. It can be written as

$$p(t, \dot{\theta}) = \begin{cases} \bar{k}_+\delta(t) & \text{if } \dot{\theta} > 0, \\ \bar{k}_-\delta(t) & \text{if } \dot{\theta} < 0, \end{cases} \tag{2}$$