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Limit cycles for m-piecewise discontinuous polynomial Liénard differential equations

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Abstract. We provide lower bounds for the maximum number of limit cycles for the m-piecewise discontinuous polynomial differential equations $\dot{x} = y + \text{sgn}(g_m(x, y))F(x)$, $\dot{y} = -x$, where the zero set of the function $\text{sgn}(g_m(x, y))$ with $m = 2, 4, 6, \ldots$ is the product of $m/2$ straight lines passing through the origin of coordinates dividing the plane into sectors of angle $2\pi/m$, and $\text{sgn}(z)$ denotes the sign function.

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1. Introduction and statement of the main result

The discontinuous Lienard polynomial differential systems have many applications, see for instance the excellent paper of Makarenkov and Lamb [23]. As far as we know up to now, there are no papers studying their limit cycles. The main objective of this paper is to start this study.

Hilbert [14] in 1900 and in the second part of its 16th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial differential systems of a given degree and also to study their distribution or configuration in the plane. Except for the Riemann hypothesis, the 16th problem seems to be the most elusive of Hilbert's problems. It has been one of the main problems in the qualitative theory of planar differential equations in the XX century. The contributions of Écalle [11] and Ilyashenko [15] proving that any polynomial differential system has finitely many limit cycles have been the best results in this area. But until now it is not proved the existence of a uniform upper bound. This problem remains open even for the quadratic polynomial differential systems. However, it is not difficult to see that any finite configuration of limit cycles is realizable for some polynomial differential system, see for details [21].

Thus, we have the finiteness of the number of limit cycles for every polynomial differential system of degree n, but we do not have uniform bounds for that number in the whole class of all polynomial differential systems of degree n. Following Smale [25], we consider an easier and special class of polynomial differential systems, the *Liénard polynomial differential systems*:

$$\begin{aligned} \dot{x} &= y + F(x), \\ \dot{y} &= -x, \end{aligned} \tag{1}$$

where $F(x) = a_0 + a_1x + \cdots + a_nx^n$, and the dot denotes derivative with respect to the time t. For these systems, the existence of uniform bounds also remains unproved. But when the degree n of these systems

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