LIMIT CYCLES BIFURCATING FROM A k–DIMENSIONAL ISOCHRONOUS CENTER CONTAINED IN $\mathbb{R}^n$ WITH $k \leq n$

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Abstract. The goal of this paper is double. First, we illustrate a method for studying the bifurcation of limit cycles from the continuum periodic orbits of a k–dimensional isochronous center contained in $\mathbb{R}^n$ with $n \geq k$, when we perturb it in a class of $\mathcal{C}^2$ differential systems. The method is based in the averaging theory.

Second, we consider a particular polynomial differential system in the plane having a center and a non–rational first integral. Then we study the bifurcation of limit cycles from the periodic orbits of this center when we perturb it in the class of all polynomial differential systems of a given degree. As far as we know this is one of the first examples that this study can be made for a polynomial differential system having a center and a non–rational first integral.

1. Introduction and statement of the results

We say that a singular point $p \in \mathbb{R}^n$ of a differential system is a *k–dimensional center* if there exists a k–dimensional submanifold M of $\mathbb{R}^n$ with $k \leq n$ such that $p \in M$, M is invariant under the flow of the differential system, and all the orbits $M \setminus \{p\}$ are periodic. Moreover, we say that the k–dimensional center p is *isochronous* if all its periodic orbit have the same period.

In the first part of this paper we illustrate a method for studying the limit cycles bifurcating from the periodic orbits of a k–dimensional isochronous center contained in $\mathbb{R}^n$ with $k \leq n$, by studying with all the details an example with $k = 2$ and $n = 4$.

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In recent years equations of the form

$$x^{IV} + bx^{II} + ax = \psi\left(t, x, x^{I}, x^{II}, x^{III}\right)$$

arise in many contexts. For example, the simplest cases when $\psi = x^2$ and $\psi = x^3$ describe the travelling waves solutions of some Korteweg–de Vries equations (KdV) and nonlinear Schrödinger equations, see [3, 8, 9]. On the other hand, the existence of periodic solutions is discussed in [4] for equations modeling undamped oscillators and having the form $x^{II} + \omega_0^2 x = \phi(t)$ where $\omega_0 > 0$, and ϕ is a continuous periodic function whose period is normalized to 2π. In this paper we deal with a particular differential equation of order four of the form

$$\frac{d^4x}{dt^4} + \alpha\, x + \psi(x,t) = 0.$$

This class of equations have been studied in [10, 12]. Here we will analyze the particular differential equation

$$\frac{d^4x}{dt^4} - x - \varepsilon\sin(x+t) = 0,$$

or equivalently the differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= x + \varepsilon\sin(x+t), \end{aligned} \tag{1}$$

where the dot denotes the derivative with respect to the time variable t. Our main result is the content of the following theorem.

Theorem 1. *For $|\varepsilon| \neq 0$ sufficiently small the differential system* (1) *has an arbitrary number of limit cycles bifurcating from the continuum of the periodic orbits of the 2–dimensional isochronous center that the system has for $\varepsilon = 0$.*

The proof of Theorem 1 is given in Section 3, and uses the averaging theory, more precisely the proof uses Theorem 3.

In the second part we deal with the homogeneous polynomial differential system

$$\begin{aligned} \dot{x} &= -y(3x^2+y^2), \\ \dot{y} &= x(x^2-y^2), \end{aligned} \tag{2}$$

of degree 3 that has the non–rational first integral

$$H(x,y) = (x^2+y^2)\exp\left(-\frac{2x^2}{x^2+y^2}\right).$$

Theorem 2. *The homogeneous polynomial differential system* (2) *has a global center at the origin (i.e. all the orbits contained in* $\mathbb{R}^2 \setminus \{(0,0)\}$ *are periodic). Let* $P(x,y)$ *and* $Q(x,y)$ *be two polynomials of degree at most* m*. Then, for convenient polynomials* P *and* Q*, the polynomial differential system*

$$(3) \qquad \begin{array}{rcl} \dot{x} & = & -y(3x^2+y^2)+\varepsilon P(x,y), \\ \dot{y} & = & x(x^2-y^2)+\varepsilon Q(x,y), \end{array}$$

has $[(m-1)/2]$ *limit cycles bifurcating from the periodic orbits of the global center* (2)*, where* $[\cdot]$ *denotes the integer part function.*

As far as we know this is one of the first examples for which the limit cycles bifurcating from the periodic orbits of a 2–dimensional center of a polynomial differential system having a non–rational first integral have been studied. The unique other example that we know was given recently in [6].

The proof of Theorem 2 is given in Section 4. Again we use averaging. More precisely, we will apply Theorem 4, which gives a method to determine bifurcation of periodic solutions from isochronous centers. We note that the center of system (2) is not isochronous; but, after a change of variables, it can be transformed to an isochronous center

We also show in Section 4 that Theorem 2 can be proved using the theory based on the generalized Abelian integrals, see a definition of these integrals at the end of Section 2.

2. Basic results

In this section first we present the basic results from the averaging theory and Abelian integrals that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T–periodic solutions from the differential system

$$(4) \qquad \mathbf{x}'(t) = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon),$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here, the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$(5) \qquad \mathbf{x}'(t) = F_0(t,\mathbf{x}),$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory. For a general introduction to the averaging theory see the books of Sanders and Verhulst [13], and of Verhulst [14].

Let $\mathbf{x}(t,\mathbf{z})$ be the solution of the unperturbed system (5) such that $\mathbf{x}(0,\mathbf{z}) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t,\mathbf{z})$ as

$$\mathbf{y}' = D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z}))\mathbf{y}. \tag{6}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (6), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1,\ldots,x_n) = (x_1,\ldots,x_k)$.

Theorem 3. *Let $V \subset \mathbb{R}^k$ be open and bounded, and let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha))\,,\ \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t,\mathbf{z}_\alpha)$ of (5) is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (6) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$ has in the upper right corner the $k\times(n-k)$ zero matrix, and in the lower right corner a $(n-k)\times(n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.*

We consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi\left(\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t)F_1(t,\mathbf{x}(t,\mathbf{z}_\alpha))dt\right). \tag{7}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left((d\mathcal{F}/d\alpha)(a)\right) \neq 0$, then there is a T–periodic solution $\varphi(t,\varepsilon)$ of system (4) such that $\varphi(0,\varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 3 goes back to Malkin [7] and Roseau [11], for a shorter proof see [2].

We assume that there exists an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic, where $\mathbf{x}(t,\mathbf{z},0)$ denotes the solution of the unperturbed system (5) with $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z}$. The set $\mathrm{Cl}(V)$ is *isochronous* for the system (4); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T–periodic solutions from the periodic solutions $\mathbf{x}(t,\mathbf{z},0)$ contained in $\mathrm{Cl}(V)$ is given in the following result.

Theorem 4. (Perturbations of an isochronous set) *We assume that there exists an open and bounded set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, the solution $\mathbf{x}(t,\mathbf{z})$ is T–periodic, then we consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \int_0^T M^{-1}_{\mathbf{z}}(t,\mathbf{z})F_1(t,\mathbf{x}(t,\mathbf{z}))dt. \tag{8}$$

If there exists $a \in V$ *with* $\mathcal{F}(a) = 0$ *and* $\det\left(\left(d\mathcal{F}/d\mathbf{z}\right)(a)\right) \neq 0$*, then there exists a* T*–periodic solution* $\varphi(t,\varepsilon)$ *of system* (4) *such that* $\varphi(0,\varepsilon) \to a$ *as* $\varepsilon \to 0$.

For a proof of Theorem 4 see Corollary 1 of [2].

Now we summarize the results on generalized Abelian integrals that we shall use.

Suppose that the unperturbed system

$$
\begin{aligned}
\dot{x} &= f(x,y), \\
\dot{y} &= g(x,y),
\end{aligned}
\tag{9}
$$

has a first integral $H(x,y)$ with an integrating factor $1/R(x,y)$. Assume that the origin of this system is a center and that the periodic orbits of this center are given by the family of ovals γ_h contained in the level curves $\{H(x,y) = h\}$.

Now we consider the perturbed system

$$
\begin{aligned}
\dot{x} &= f(x,y) + \varepsilon P(x,y), \\
\dot{y} &= g(x,y) + \varepsilon Q(x,y),
\end{aligned}
$$

which can be written into the form

$$
\begin{aligned}
\dot{x} &= -\frac{\partial H}{\partial y}(x,y)R(x,y) + \varepsilon P(x,y), \\
\dot{y} &= \frac{\partial H}{\partial x}(x,y)R(x,y) + \varepsilon Q(x,y).
\end{aligned}
\tag{10}
$$

Then the generalized Abelian integral associated to this system is

$$
I(h) = \int_{\gamma_h} \frac{P(x,y)dy - Q(x,y)dx}{R(x,y)}.
\tag{11}
$$

Since $I(h)$ gives the first order approximation in ε of the displacement function, we get the following result.

Theorem 5. *The simple zeros of the function* $I(h)$ *provide limit cycles for the perturbed system* (10) *which bifurcate from the periodic orbits of the unperturbed system* (9).

For more details about (generalized) Abelian integrals and the proof of Theorem 5 see Li [5].

3. Perturbation of a 2–dimensional isochronous center in $\mathbb{R}^4$

In this section we prove Theorem 1.

The linear part at the origin of the differential system (1) is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

and its eigenvalues are ± 1 and $\pm i$. Doing the change of variables $(x, y, z, w) \mapsto (X, Y, Z, W)$ given by

$$\begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

system (1) becomes

$$\begin{array}{rcl} \dot{X} & = & -Y + \varepsilon \sin((4t + X - Y + Z - W)/4), \\ \dot{Y} & = & X + \varepsilon \sin((4t + X - Y + Z - W)/4), \\ \dot{Z} & = & Z + \varepsilon \sin((4t + X - Y + Z - W)/4), \\ \dot{W} & = & -W + \varepsilon \sin((4t + X - Y + Z - W)/4). \end{array} \tag{13}$$

Note that the differential of this system at the origin is the real normal Jordan form of the matrix (12).

Now we shall apply Theorem 3 to the differential system (13) taking

$$\begin{array}{l} \mathbf{x} = (X, Y, Z, W), \\ F_0(t, \mathbf{x}) = (-Y, X, Z, -W), \\ F_1(t, \mathbf{x}) = (A, A, A, A), \\ F_2(t, \mathbf{x}, \varepsilon) = 0, \\ \Omega = \mathbb{R}^4, \end{array} \tag{14}$$

where $A = \sin((4t + X - Y + Z - W)/4)$.

Clearly system (13) with $\varepsilon = 0$ has a linear center at the origin in the (X, Y)–plane. We remark that all linear centers are isochronous. Using the notation of Section 2 (mainly the notation related with the statement of Theorem 3), the periodic solution $\mathbf{x}(t, \mathbf{z})$ of this center with $\mathbf{z} = (X_0, Y_0, 0, 0)$ is

$$\begin{array}{rcl} X(t) & = & X_0 \cos t - Y_0 \sin t, \\ Y(t) & = & Y_0 \cos t + X_0 \sin t, \\ Z(t) & = & 0, \\ W(t) & = & 0, \end{array} \tag{15}$$

with period $T = 2\pi$. The V and α of Theorem 3 are

$$V = \{(X, Y, 0, 0) : 0 < X^2 + Y^2 < \rho\},$$

for some real number $\rho > 0$, and $\alpha = (X_0, Y_0) \in V$.

For the function F_0 given in (14) and the periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ given in (15) the fundamental matrix $M(t)$ of the differential system (6) such that $M(0)$ is the identity matrix of $\mathbb{R}^4$ is

$$M(t) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}.$$

We remark that for system (13) with $\varepsilon = 0$ the fundamental matrix does not depend on the particular periodic orbit $\mathbf{x}(t, \mathbf{z})$; i.e. it is independent of the initial conditions $\mathbf{z}$. Therefore, an easy computation shows that

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - e^{-2\pi} & 0 \\ 0 & 0 & 0 & 1 - e^{2\pi} \end{pmatrix}.$$

Consequently all the assumptions of Theorem 3 are satisfied. Therefore, we must study the zeros in V of the system $\mathcal{F}(\alpha) = 0$ of two equations and two unknowns, where $\mathcal{F}$ is given by (7). More precisely, we have $\mathcal{F}(\alpha) = (\mathcal{F}_1(X_0, Y_0), \mathcal{F}_2(X_0, Y_0))$ where

$$\mathcal{F}_1 = \int_0^{2\pi} (\cos t + \sin t) \sin \left[t + \frac{(X_0 - Y_0)\cos t - (X_0 + Y_0)\sin t}{4} \right] dt,$$
$$\mathcal{F}_2 = \int_0^{2\pi} (\cos t - \sin t) \sin \left[t + \frac{(X_0 - Y_0)\cos t - (X_0 + Y_0)\sin t}{4} \right] dt.$$

After a tedious calculation (which can be checked using an algebraic processor) and the change of variables $(X_0, Y_0) \mapsto (r, s)$ given by

$$\begin{array}{rcl} X_0 - Y_0 & = & 4r \cos s, \\ X_0 + Y_0 & = & -4r \sin s, \end{array}$$

we obtain for

$$g_j(r, s) = \mathcal{F}_j(2r(\cos s - \sin s), -2r(\cos s + \sin s)),$$

with $j = 1, 2$, that

$$\begin{array}{rcl} g_1(r,s) & = & \pi[J_0(r) + J_2(r)(\cos 2s - \sin 2s)], \\ g_2(r,s) & = & -\pi[J_0(r) + J_2(r)(\cos 2s + \sin 2s)], \end{array}$$

where $J_{\mu(r)}$ is the μ–th Bessel function of first kind (see [1]).

Adding and subtracting the two equations $g_j(r,s) = 0$, for $j = 1, 2$, we obtain the system

$$\begin{array}{rcl} h_1(r,s) & = & J_2(r)\sin 2s = 0, \\ h_2(r,s) & = & J_0(r) + J_2(r)\cos 2s = 0. \end{array} \tag{16}$$

It is known that the zeros of the functions $J_\mu(r)$ are distinct for different μ's, then either $s = 0$, or $s = \pi/2$. We are not interested in all the solutions of this system, we are only interested to show that it has as many solutions as we want satisfying the assumptions of Theorem 3. So, in what follows we only study the solutions with $s = 0$. Consequently, from the second equation of (16) we obtain

$$J_0(r) + J_2(r) = 0.$$

Since $J_0(r) + J_2(r) = 2J_1(r)/r$, and the function $J_1(r)$ has infinitely many positive zeros tending to be uniformly distributed when $r \to \infty$, because the asymptotic behavior of $J_1(r)$ is $\sqrt{2/(\pi r)}\cos(r - 3\pi/4)$, it follows that system (16) has infinitely many solutions of the form $(r_0, 0)$ being r_0 a positive zero of $J_1(r)$. Then, $(X_0, Y_0) = (2r_0, -2r_0)$ is a solution of the system $\mathcal{F}_j(X_0, Y_0) = 0$ for $j = 1, 2$. Moreover, the determinant of $\partial(\mathcal{F}_1, \mathcal{F}_2)/\partial(X_0, Y_0)$ at the point $(2r_0, -2r_0)$ is

$$\det(r_0) = \frac{\pi^2}{8} r_0^2 \mathcal{H}(3, -r_0^2/2)\left[\mathcal{H}(3, -r_0^2/2) - \mathcal{H}(2, -r_0^2/2)\right].$$

where $\mathcal{H}$ is the regularized hypergeometric function, see [1]. Using the formula

$$J_\mu(z) = \frac{z^2}{2^\mu(\mu+1)!}\,\mathcal{H}(\mu+1, -z^2/4),$$

we get

$$\det(r_0) = \frac{72\,\pi^2 J_2(r_0)^2}{r_0^2}.$$

Since the zeros of $J_1(r)$ and $J_2(r)$ are different, we get that $\det(r_0) \neq 0$. Hence, by Theorem 3 for each $(2r_0, -2r_0)$ contained in V we have a periodic orbit of system (13) with $|\varepsilon| \neq 0$ sufficiently small.

Finally, for a given positive integer N we can fix ρ in the definition of V in such a way that the interval $(0, \rho)$ contains exactly N zeros of the function $J_1(r)$. Then taking $|\varepsilon| \neq 0$ small enough, Theorem 3 guarantees the existence of N periodic orbits for system (13). Moreover, choosing $|\varepsilon| \neq 0$ smaller if necessary, since system (13) with $\varepsilon = 0$ has its periodic orbits strongly stable and unstable in the directions Z and W respectively, it follows that the N periodic orbits for system (13) obtained using Theorem 3 are limit cycles; i.e. they are isolated in the set of all periodic orbits. This completes the proof of Theorem 1.

4. Perturbation of a 2–dimensional center having a non–rational first integral

First we show that the homogeneous polynomial differential system (2) has a global center at the origin. In polar coordinates (r, θ) defined by $x = r\cos\theta$, $y = r\sin\theta$, system (2) becomes

$$\begin{array}{rcl} \dot{r} & = & -r^3 \sin 2\theta, \\ \dot{\theta} & = & r^2. \end{array}$$

Of course, to study this system is equivalent to study the differential equation

$$\frac{dr}{d\theta} = -r\sin 2\theta, \tag{17}$$

whose solution $r(\theta, z)$ satisfying $r(0, z) = z$ is

$$r(\theta, z) = z\exp\left(-\sin^2\theta\right). \tag{18}$$

Therefore all the solutions of the differential equation (17) and consequently all the solutions of the homogeneous polynomial differential system (2) are periodic with the exception of the origin which is a singular point. Hence it is proved that the origin of system is a global center.

Now we want to study the limit cycles of the perturbed system (3) for $|\varepsilon| \neq 0$ sufficiently small, which bifurcate from the periodic orbits of the center of system (2).

We write the polynomial $P(x, y)$ of degree m of system (2) as

$$P(x, y) = \sum_{l=0}^{m} P_l(x, y),$$

where $P_l(x, y)$ is the homogeneous part of degree l of $P(x, y)$. We do the same for the polynomial $Q(x, y)$.

Doing for system (13) the same changes of variables that we have done to system (2), we obtain that system (2) can be written as

$$\frac{dr}{d\theta} = -r\sin 2\theta + \varepsilon F_1(\theta, r) + O(\varepsilon^2), \tag{19}$$

where

$$F_1(\theta, r) = \sum_{l=0}^{m} r^l - 2 \quad [(\cos\theta + \cos 3\theta)P_l(\cos\theta, \sin\theta)+ \\ (3\sin\theta + \sin 3\theta)Q_l(\cos\theta, \sin\theta)].$$

Now we shall apply Theorem 4 to the differential equation (19) taking $k = n = 1$ and

$$\begin{array}{l} \mathbf{x} = r, \\ t = \theta, \\ F_0(\theta, \mathbf{x}) = -r\sin 2\theta, \\ \Omega = (0, \infty). \end{array} \tag{20}$$

Clearly the differential equation (19) is $T = 2\pi$ periodic in the variable θ. Moreover this equation for $\varepsilon = 0$ has all its solutions 2π–periodic and given by (18). The V and α of Theorem 4 are

$$V = \{r : 0 < r < \rho\},$$

for some real number $\rho > 0$, and $\alpha = z \in V$.

For the function F_0 given in (20) and the periodic solution $r(\theta, z)$ given in (18) the 1×1 fundamental matrix $M(\theta)$ of the differential equation (19) with $\varepsilon = 0$ such that $M(0) = (1)$ is

$$M(\theta) = (e^{-\sin^2\theta}).$$

We remark that for system (20) the fundamental matrix does not depend on the particular periodic orbit $r(\theta, z)$; i.e. it is independent of the initial condition z. Therefore

$$M^{-1}(\theta) = (e^{\sin^2\theta}).$$

Since all the assumptions of Theorem 4 are satisfied, we must study the zeros in V of the function $\mathcal{F}(z)$, where $\mathcal{F}$ is given by (8). More precisely, we have

$$\mathcal{F}(z) = \sum_{l=0}^{m} z^{l-2} I_l,$$

where

$$I_l = \int_0^{2\pi} e^{(3-l)\sin^2\theta} \quad [(\cos\theta + \cos 3\theta)P_l(\cos\theta, \sin\theta)+ \\ (3\sin\theta + \sin 3\theta)Q_l(\cos\theta, \sin\theta)]d\theta.$$

By symmetry the integral $I_l = 0$ if l is even. So, if $m = 2\nu + 1$ then

$$\mathcal{F}(z) = \frac{1}{z} \sum_{l=0}^{\nu} z^{2l} I_{2l+1}. \tag{21}$$

Hence the polynomial $\mathcal{F}(z)$ at most can have $\nu = [(m-1)/2]$ positive real roots. If $m = 2\nu$ then

$$\mathcal{F}(z) = \frac{1}{z}\sum_{l=0}^{\nu-1} z^{2l} I_{2l+1}. \tag{22}$$

Therefore, again the polynomial $f(z)$ at most can have $\nu - 1 = [(m-1)/2]$ positive real roots. In short, using Theorem 4 we at most can get $[(m-1)/2]$ limit cycles of system (3) bifurcating from the periodic orbits of system (2).

We shall prove that when $m = 2\nu + 1$ the function (21) for the perturbed system (3) with

$$P(x,y) = \sum_{l=0}^{\nu} a_{2l+1} x^{2l+1}, \quad Q(x,y) = 0, \tag{23}$$

can be chosen in order that it has exactly $\nu = [(m-1)/2]$ positive arbitrary zeros. In a similar way it can be proved that when $m = 2\nu$ the function (22) for the perturbed system (3) with

$$P(x,y) = \sum_{l=0}^{\nu-1} a_{2l+1} x^{2l+1}, \quad Q(x,y) = 0,$$

can be chosen in order that it has exactly $\nu - 1 = [(n-1)/2]$ positive arbitrary zeros. Hence Theorem 4 will be proved.

Now for $m = 2\nu + 1$ we consider system (3) with P and Q given by (23). Then for its corresponding function (21) we have

$$I_{2l+1} = a_{2l+1} \int_0^{2\pi} e^{2(1-l)\sin^2\theta} (\cos\theta + \cos 3\theta) \cos^{2l+1}\theta \, d\theta.$$

Again after a tedious computation (that we can help with an algebraic manipulator as mathematica or maple) we obtain that I_{2l+1} is equal to

$$\begin{aligned} L(l) = \; & (12l^2 - 28l + 19)\, \overline{\mathcal{H}}(l + 1/2, l + 3, 2l - 2) \\ & + (l+2)(6l-5)\, \overline{\mathcal{H}}(l - 1/2, l + 2, 2l - 2), \end{aligned} \tag{24}$$

multiplied by the constant

$$\frac{2L - 1}{(l+2)!} \sqrt{\pi}\, \Gamma(l - 1/2)\, e^{2-2l},$$

where $\overline{\mathcal{H}}(a, b, z)$ is the Kummer confluent hypergeometric function and $\Gamma(z)$ is the Gamma function, see [1]. The value of $L(l)$ is non–zero for all non–negative integer l, see the appendix. Hence in the polynomial (21) we always can choose the coefficients a_{l+1} conveniently and alternating the sign (by the Descartes rule) in order that the polynomial has the maximum possible number of positive roots $\nu = [(m-1)/2]$. This completes the proof of Theorem 2.

Finally we remark that if we compute the generalized Abelian integral (11) for the system (3) taking

$$x(\theta) = ze^{-\sin^2\theta}\cos\theta, \quad y(\theta) = ze^{-\sin^2\theta}\sin\theta,$$

as a parametrization of the center when $\varepsilon = 0$ and integrating with respect to the variable θ between 0 and 2π, we obtain that

$$I(z) = \frac{z}{e^2}\,\mathcal{F}(z).$$

Hence both functions have the same positive zeros, and in this case the method based in Theorem 3 and the method based in the generalized Abelian integral coincide.

5. The appendix

The Kummer confluent hypergeometric function is given by the series

$$\overline{\mathcal{H}}(a,b,z) = \sum_{\mu=0}^{\infty} \frac{\Gamma(a+\mu)\Gamma(b)}{\Gamma(a)\Gamma(b+\mu)\mu!}\, z^{\mu}. \tag{25}$$

We shall prove that $L(l)$ defined in (24) is always positive for any integer $l \geq 0$. For showing that we will compute the coefficients of the series expansion of $L(l)$ and we will see that all of them are positive.

Using (25) we obtain that

$$L(l) = \sum_{\mu=0}^{\infty} L_{\mu,l}\,(2l-2)^{\mu},$$

where the coefficient $L_{\mu,l}$ is equal to

$$\frac{(36l^2-72l+43)\mu + 3(2l-1)(6l^2-7l+3)(l+2)!\,\Gamma(l+\mu-1/2)}{\Gamma(l+1/2)(l+\mu+2)!\,\mu!}.$$

For $l > 0$ since $36l^2 - 72l + 43 > 0$, $6l^2 - 7l + 3 > 0$, $\Gamma(l+\mu-1/2) > 0$ and $\Gamma(l+1/2) > 0$ it follows that $L_{\mu,l} > 0$ for $\mu = 0, 1, 2\ldots$ and $l > 0$. Therefore $L(l) > 0$ if $l > 0$.

Finally, from (25) we have

$$L(0) = 2e\pi(J_0(1) - 2J_1(1)) \approx 2.31849804 > 0.$$

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