ON THE MAXIMUM NUMBER OF LIMIT CYCLES OF DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS WITH A STRAIGHT LINE OF SEPARATION

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Abstract. In this paper we study the maximum number of limit cycles for planar discontinuous piecewise linear differential systems defined in two half–planes separated by a straight line. Here we only consider non-sliding limit cycles. For that systems, the interior of any limit cycle only contains a unique singular point or a unique sliding segment. Moreover, the linear differential systems that we consider in every half–plane can have either a focus (F), or a node (N), or a saddle (S), these equilibrium points can be real or virtual. Then, we can consider six kinds of planar discontinuous piecewise linear differential systems: FF, FN, FS, NN, NS, SS. We analyze for each of these types of discontinuous differential systems the maximum number of known limit cycles.

1. Introduction and statement of the main results

The study of piecewise linear differential systems goes back to Andronov and coworkers [2], and today still are studied by many researchers. Thus, these last years a big interest from the mathematical community takes place trying to understand their dynamical richness, because such systems are widely used to model many real processes and different modern devices, see for more details [6] and the references therein. More recently, these systems become also relevant as idealized models of cell activity, see [4, 20, 21].

The case of continuous piecewise linear differential systems, when they have only two half–planes separated by a straight line is the simplest possible configuration of piecewise linear differential systems. In 1990 Lum and Chua [17] conjectured that a continuous piecewise linear vector field in the plane with two zones has at most one limit cycle. In 1998 this conjecture was proved by Freire, Ponce, Rodrigo and Torres in [9]. We note that even in this apparent simple case, only after a hard analysis it was possible to establish the existence of at most one limit cycle, see [9]. The reason for these difficulties in the continuous piecewise linear differential systems is double. First, even we can easily compute their solutions in every half–plane, in general the time that each orbit requires to pass from one half–plane to the other is not known explicitly and consequently the matching of the corresponding solutions is a complicate problem. Second, in general the number of parameters to consider in order to take into account all possible configurations is not small.

The purpose of this paper is to study the problem of Lum and Chua for the class of discontinuous piecewise linear vector field in the plane with two zones.

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Limit cycles of discontinuous piecewise linear differential systems defined on two half–planes separated by a straight line have been studied recently in [3, 10, 11, 12, 13, 14], among other papers. The linear differential systems that we consider in every half–plane can be in the full plane either a focus (F), or a node (N), or a saddle (S), of course all these equilibrium points are hyperbolic, i.e. with eigenvalues having their real part different from zero. We note that there are three classes of hyperbolic linear nodes: nodes with different eigenvalues, nodes with equal eigenvalues whose linear part does not diagonalize, and nodes with equal eigenvalues whose linear diagonalize. Clearly this last class of linear nodes have all the straight lines through the node invariant and consequently the corresponding piecewise linear differential systems cannot have periodic orbits.

We say that the equilibrium point of a given linear differential system defined in a half–plane having in the full plane a focus, a node or a saddle is *real* when this equilibrium point belongs to the closure of the half–plane where we have defined the corresponding linear differential system, and it is called *virtual* otherwise.

In short we distinguish six classes or types of planar discontinuous piecewise linear differential systems: FF, FN, FS, NN, NS, SS. Inside these classes we only consider limit cycles surrounding a unique equilibrium point or a unique sliding segment. For a definition of sliding segment see for instance [19]. Moreover, it is worth saying that we do not consider the so called sliding limit cycles, i.e. limit cycles containing a sliding segment, see [10] for more details on these kind of limit cycles.

In this paper and in what follows the discontinuous piecewise linear differential systems, when they have only two half–planes separated by a straight line and are of one of the above six classes will be denoted simply by *discontinuous systems*.

Han and Zhang provided discontinuous systems of the types FF, FN and NN with two limit cycles, and they conjectured that the maximum number of limit cycles for this class of discontinuous systems is exactly two, see [11]. However, by considering a special family of discontinuous systems sharing the equilibrium position, Huan and Yang in [12] provided strong numerical evidence about the existence of three limit cycles in the FF case. The example in [12] represents up to the best of our knowledge the first discontinuous system with 3 limit cycles surrounding a unique equilibrium. Later on Llibre and Ponce in [14] provided a proof of the existence of such 3 limit cycles. Also recently Artés, Llibre, Medrado and Teixeira in [3] show that the type SS can exhibit 2 limit cycles.

In summary, the maximum number of limit cycles of discontinuous systems up to now are given in Table 1(a). In that table the symbol – indicates that those cases appear repeated in the table, because for instance the case NF is the same that the case FN, which already appears with a 2 in the table.

In this paper we will prove that in the cases FS and NS there are discontinuous systems having 2 limit cycles. These two limit cycles bifurcates from a homoclinic connection. So, the maximum number of limit cycles of discontinuous systems up to now are given in Table 1(b). Thus our main results are the following three theorems.

	F	N	S
F	3	2	?
N	–	2	?
S	–	–	2

(a)

	F	N	S
F	3	2	2
N	–	2	2
S	–	–	2

(b)

TABLE 1. The maximum number of known limit cycles of discontinuous piecewise linear differential systems with a straight line of separation: (a) before this paper and (b) from this paper.

Theorem 1 (FS case). *For $0 < \delta \ll \varepsilon \ll 1$ the following planar discontinuous piecewise linear differential system given by*

$$\begin{aligned} \dot{x} &= \frac{1-\delta}{\pi}\log\left(\frac{11(5-3\varepsilon)}{14(3\varepsilon+5)}\right)x+y, \\ \dot{y} &= -x+\frac{1-\delta}{\pi}\log\left(\frac{11(5-3\varepsilon)}{14(3\varepsilon+5)}\right)y, \end{aligned} \tag{1}$$

if $x>0$, and by

$$\begin{aligned} \dot{x} &= \frac{3}{2}(1+\varepsilon)x+y-\frac{3}{10}, \\ \dot{y} &= 4x-\frac{3}{2}(1-\varepsilon)y+\frac{67}{10}, \end{aligned} \tag{2}$$

if $x<0$, is of type FS and has 2 limit cycles.

Theorem 2 (NS case, here the linear part of the node does not diagonalize). *For $-1 \ll \varepsilon \ll \delta < 0$ sufficiently small the following planar discontinuous piecewise linear differential system given by*

$$\begin{aligned} \dot{x} &= -\left(\sqrt{3}+\log\left(2-\sqrt{3}\right)(1+\delta)\right)x+y, \\ \dot{y} &= -3x+\left(\sqrt{3}-\log\left(2-\sqrt{3}\right)(1+\delta)\right)y, \end{aligned} \tag{3}$$

if $x>1$, and by

$$\begin{aligned} \dot{x} &= \varepsilon x+y, \\ \dot{y} &= x+\varepsilon y, \end{aligned} \tag{4}$$

if $x<1$, is of type NS and has 2 limit cycles.

Theorem 3 (NS case, here the linear part of the node diagonalize but the eigenvalues are not equal). *For suitable values of ε and δ satisfying $0<\delta\ll\varepsilon\ll 1$ the following planar discontinuous piecewise linear differential system given by*

$$\begin{aligned} \dot{x} &= -x+y, \\ \dot{y} &= -2y-4, \end{aligned} \tag{5}$$

if $x>0$, and by

$$\begin{aligned} \dot{x} &= (\varepsilon-1)x+2y+\varepsilon-\delta-1, \\ \dot{y} &= 4x+(\varepsilon+1)y-2\delta+4, \end{aligned} \tag{6}$$

if $x<0$, is of type NS and has 2 limit cycles.

Theorems 1, 2 and 3 are proved in sections 2, 3 and 4, respectively. In section 5 we illustrate the above results computing numerically these limit cycles. Moreover, we show how the parameter δ is much more small than ε. In all cases the computations have been very accurate in order to distinguish both limit cycles. When the corresponding system has a node, Theorem 3, to find the values of δ when ϵ decreases is much more difficult. Because the speed of the solutions of the vector field in this side is very high and these solutions remain a very short time in this side.

2. Proof of Theorem 1

Since $0 < \delta \ll \varepsilon \ll 1$ clearly system (1) has a stable focus at the origin of coordinates because its linear part has eigenvalues

$$(1-\delta)\log\left(\frac{11(5-3\varepsilon)}{14(3\varepsilon+5)}\right) \pm i,$$

for more details see for instance Theorem 2.15 of [7]. On the other hand system (2) has a saddle at the equilibrium point

$$\left(\frac{9\varepsilon+125}{5(9\varepsilon^2-25)}, -\frac{3(67\varepsilon+75)}{5(9\varepsilon^2-25)}\right)$$

with eigenvalues $(3\varepsilon \pm 5)/2$, the separatrices of this saddle live on the invariant straight lines $y = x + 14/(5-3\varepsilon)$ and $y = -4x - 11/(5+3\varepsilon)$. On the first straight line live the two unstable separatrices and in the second one the two stable ones. Again for more details see Theorem 2.15 of [7]. See Figure 1.

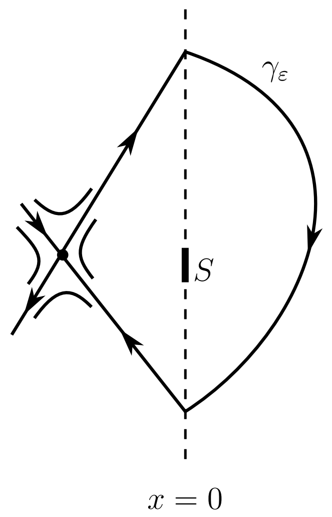

Figure 1. The saddle-loop γ_ε.

Since the differential systems (1) and (2) are linear we can explicitely compute their solutions. We denote by $(x_+(t,y_0), y_+(t,y_0))$ the solution of system (1) such that $(x_+(0,y_0), y_+(0,y_0)) = (0,y_0)$, and by $(x_-(t,y_0), y_-(t,y_0))$ the solution of system (2) such that $(x_-(0,y_0), y_-(0,y_0)) = (0,y_0)$. Then we have

$$x_+(t,y_0) = \left(\frac{55-33\varepsilon}{70+42\varepsilon}\right)^{\frac{t-t\delta}{\pi}} y_0 \sin t, \quad y_+(t,y_0) = \left(\frac{55-33\varepsilon}{70+42\varepsilon}\right)^{\frac{t-t\delta}{\pi}} y_0 \cos t,$$

and

$$\begin{aligned} x_-(t,y_0) &= \frac{e^{\frac{1}{2}t(3\varepsilon+5)}}{5\,(9\varepsilon^2-25)}\Big(\,(9\varepsilon+125)\,e^{-\frac{1}{2}(3\varepsilon+5)t} \\ &\quad -(3\varepsilon+5)\,((3\varepsilon-5)y_0+14)\,e^{-5t}+(3\varepsilon-5)\,((3\varepsilon+5)y_0+11)\,\Big), \\ y_-(t,y_0) &= \frac{e^{\frac{1}{2}t(3\varepsilon+5)}}{5\,(9\varepsilon^2-25)}\Big(-3\,(67\varepsilon+75)\,e^{-\frac{1}{2}(3\varepsilon+5)t} \\ &\quad +4\,(3\varepsilon+5)\,((3\varepsilon-5)y_0+14)\,e^{-5t}+(3\varepsilon-5)\,((3\varepsilon+5)y_0+11)\,\Big), \end{aligned}$$

Looking at the solutions $(x_+(t,y_0),y_+(t,y_0))$ it is clear that all the orbits of system (1) starting at the point $(0,y_0)$ with $y_0>0$ intersect by first time the discontinuous straight line $x=0$ after a time equal to π. On the other hand the unstable separatrix of the saddle that intersects the discontinuous straight line $x=0$ does it at the point $(0,14/(5-3\varepsilon))$, and the stable one does it at the point $(0,-11/(5+3\varepsilon))$.

In what follows we assume that $\delta=0$, until we do not say the contrary. Then an easy computation shows that

$$y_+\left(\pi,\frac{14}{5-3\varepsilon}\right)=-\frac{11}{5+3\varepsilon}.$$

Therefore the saddle has a loop γ_ε surrounding the sliding segment S, with endpoints $(0,0)$ and $(0,3/10)$, for all $\varepsilon\in(-5/3,5/3)$, see Figure 1.

Here we follow the Filippov criteria, see [8], proved by regularization in [16, 18] and by singular perturbation theory in [15].

We claim the following two results:

(i) For $\varepsilon=0$ all the orbits in the interior of the loop γ_ε (with the exception of the sliding segment S) have as ω–limit the loop and as α–limit the sliding segment S. For a definition of α– and ω–limit see for instance [7].

(ii) For $\varepsilon>0$ sufficiently small one stable limit cycle bifurcates from the loop γ_ε, and the loop persists.

Now we prove the claim. Let $y_0\in(0,14/(5-3\varepsilon))$. Then equation $x_-(-t,y_0)=0$ provides the time $t>0$ which needs the orbit $(x_-(t,y_0),y_-(t,y_0))$ starting at the point $(0,y_0)$ when the time is zero for reaching in backward time the straight line $x=0$. From $x_-(-t,y_0)=0$ we get that

$$y_0=\frac{(9\varepsilon+125)e^{\frac{1}{2}t(3\varepsilon+5)}-(42\varepsilon+70)e^{5t}+33\varepsilon-55}{(-1+e^{5t})\,(9\varepsilon^2-25)}.$$

Substituting this y_0 in $y_-(-t,y_0)-y_+(\pi,y_0)$ we obtain

$$-\frac{e^{\frac{1}{2}t(-3\varepsilon+5)}(9\varepsilon+125)\left(11e^{3\varepsilon t}(3\varepsilon-5)+14(3\varepsilon+5)-15e^{\frac{1}{2}t(3\varepsilon-5)}(5\varepsilon+1)\right)}{14\,(-1+e^{5t})\,(3\varepsilon-5)(3\varepsilon+5)^2}.$$

In particular for $\varepsilon=0$ we have that

$$y_-(-t,y_0)-y_+(\pi,y_0)=\frac{15}{14\,(1+e^{5t/2})}>0,$$

and the claim (i) is proved, see Figure 2.

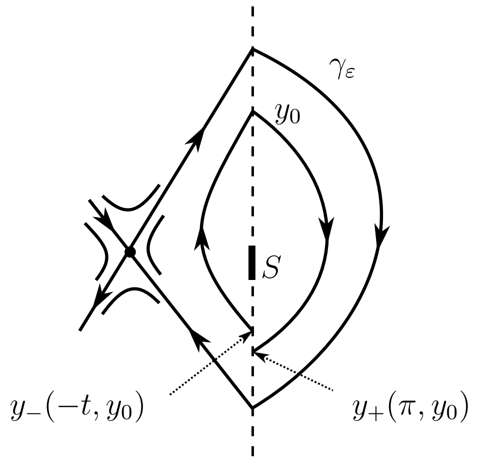

FIGURE 2. The saddle-loop γ_ε is a global attractor in its interior with exception of the sliding segment S.

We have proved that the loop γ_ε exists for $\varepsilon \in (-5/3, 5/3)$. Therefore inside the region bounded by the loop we can define a Poincaré map P_ε. Moreover, this Poincaré map is analytic, because it is a composition of two analytic Poincaré maps, one from $x = 0$ to itself following the orbits of $x > 0$, and the other from $x = 0$ to itself following the orbits of $x < 0$. Recall that the Poincaré map associated to a linear differential system is analytic because such differential system is analytic, see for more details [7].

From (i) if $\varepsilon = 0$ this Poincaré map P_0 is defined in the interior I of the segment with endpoints $(0, 0)$ and $(0, 14/5)$. From the proof of (i) we have that $P_0(y) > y$ for all $y \in I$.

Our discontinuous system (1)–(2) can be regularized as in the papers [16] or [18], in such a way that it can be obtained as a limit of smooth differential systems, and consequently the Poincaré map P_ε can also be obtained as limit of smooth Poincaré maps of smooth differential systems. Now we recall a result that can be found in the book of Andronov et al. [1] on the stability of a loop for smooth planar differential systems.

Theorem 4. *Consider a two–dimensional differential system*

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu), \quad \mathbf{x} \in \mathbb{R}^2, \quad \mu \in \mathbb{R}, \tag{7}$$

with smooth f, having at $\mu = \mu_0$ a saddle equilibrium point $\mathbf{x}_0$ with eigenvalues $\lambda_1 < 0 < \lambda_2$ and this saddle has a loop γ. Assume that $\lambda = \lambda_1 + \lambda_2 \neq 0$, then the loop γ in its interior part is unstable if $\lambda > 0$, and stable if $\lambda < 0$.

Since the Poincaré map of our discontinuous systems is limit of smooth Poincaré maps of smooth differential systems Theorem 4 also works for our discontinuous systems having the loop γ_ε. Therefore, since the trace $\lambda_1 + \lambda_2$ of our saddle is 3ε, Theorem 4 implies that for $\varepsilon > 0$ our loop γ_ε is locally unstable, but for $\varepsilon = 0$ it was stable as we have proved in (i) because all the orbits of the interior of the loop (with the exception of the sliding segment S) tend in forward time to the loop when $\varepsilon = 0$. This forces the existence of a stable limit cycle Γ_s for $\varepsilon > 0$ and sufficiently small which borns from the loop when $\varepsilon = 0$, and of course the loop remains as we have proved. This completes the proof of (ii).

Until here we have assumed that $\delta = 0$. Now if we take $0 < \delta \ll \varepsilon$ the unstable separatrix coming from the saddle until the point $(0, 14/(5-3\varepsilon))$ does not connect with the stable one at the point $(0, -11/(5+3\varepsilon))$, because

$$y_+(\pi, 14/(5-3\varepsilon)) = -\frac{11}{5+3\varepsilon}\left(\frac{55-33\varepsilon}{42\varepsilon+70}\right)^{-\delta} < -\frac{11}{5+3\varepsilon}.$$

In fact, this inequality shows that is the stable separatrix of the previous loop that now goes in backward time towards what was the interior of the region bounded by the loop before that it breaks, see Figure 3. Since inside the loop this qualitative behavior persists if $0 < \delta \ll \varepsilon$, this implies the existence of a second unstable limit cycle Γ_u which appears when the loop breaks at $\delta = 0$, and such that it contains in the region that it limits the limit cycle Γ_s. This completes the proof of Theorem 1.

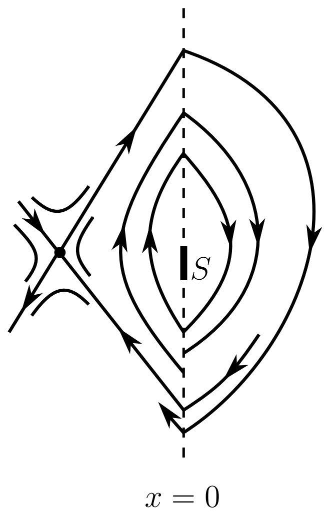

FIGURE 3. Creating the second limit cycle.

A numerical approximation of the limit cycles which born from the loop it is shown in the Appendix.

3. PROOF OF THEOREM 2

We need the following auxiliary result.

Lemma 5. *Let $W(z) = w$ be the principal part of the Lambert function which is defined implicitly by the equation $z = we^w$ (for more details see [5]). Then, the function*

$$f(t) = -\sqrt{3} - \coth t + \operatorname{csch} t + \frac{\log(2-\sqrt{3})}{W\left(\frac{(2-\sqrt{3})^{1/(\sqrt{3}-\coth t+\operatorname{csch} t)}\log(2-\sqrt{3})}{\sqrt{3}-\coth t+\operatorname{csch} t}\right)} < 0,$$

for $t > 0$.

Proof. For $t>0$ we have

$$\begin{aligned}\sqrt{3}-\coth t+\operatorname{csch} t = & \sqrt{3}-\frac{e^t-e^{-t}}{e^t-e^{-t}}+\frac{2}{e^t-e^{-t}}=\sqrt{3}+\frac{2-e^t+e^{-t}}{e^t-e^{-t}}\\ \geq & \sqrt{3}+\frac{1-e^t}{e^t-e^{-t}}=\sqrt{3}+\frac{e^t(1-e^t)}{e^{2t}-1}\\ = & \sqrt{3}+\frac{e^t(1-e^t)}{(e^t-1)(e^t+1)}=\sqrt{3}-\frac{e^t}{e^t+1}>0.\end{aligned}$$

Therefore

$$z=\frac{(2-\sqrt{3})^{1/(\sqrt{3}-\coth t+\operatorname{csch} t)}\log(2-\sqrt{3})}{\sqrt{3}-\coth t+\operatorname{csch} t}<0.$$

Since $W(z)=w$ is equivalent to $z=we^w$, we have that $z<0$ if and only if $w<0$. So $W(z)<0$. Then,

$$g(t)=\frac{\log(2-\sqrt{3})}{W\left(\frac{(2-\sqrt{3})^{1/(\sqrt{3}-\coth t+\operatorname{csch} t)}\log(2-\sqrt{3})}{\sqrt{3}-\coth t+\operatorname{csch} t}\right)}>0,$$

for $t>0$.

Tedious computations show that for $t>0$ we have that $g(t)\in(1.3,1+\sqrt{3})$, and that $-\sqrt{3}-\coth t+\operatorname{csch} t\in(-1-\sqrt{3},-\sqrt{3})$. So $f(t)<0$ when $t>0$. □

Proof of Theorem 2. Since $-1\ll\varepsilon\ll\delta<0$ clearly system (3) has a virtual unstable node at the origin of coordinates because its linear part has the eigenvalue $-(1+\delta)\log\left(2-\sqrt{3}\right)>0$ with multiplicity 2, and the real normal Jordan form of its linear part does not diagonalize, for more details see Theorem 2.15 of [7]. On the other hand system (4) has a real saddle at the origin of coordinates with eigenvalues $\varepsilon\pm1$, the separatrices of this saddle live on the invariant straight lines $y=x$ and $y=-x$. On the first straight line live the two unstable separatrices and in the second one the two stable ones. Again for more details see Theorem 2.15 of [7].

Since the differential systems (3) and (4) are linear we can explicitely compute their solutions. We denote by $(x_+(t,y_0),y_+(t,y_0))$ the solution of system (3) such that $(x_+(0,y_0),y_+(0,y_0))=(1,y_0)$, and by $(x_-(t,y_0),y_-(t,y_0))$ the solution of system (4) such that $(x_-(0,y_0),y_-(0,y_0))=(1,y_0)$. Then we have

$$\begin{aligned}x_+(t,y_0)= & \left(2-\sqrt{3}\right)^{-(1+\delta)t}\left(1+\left(y_0-\sqrt{3}\right)t\right),\\ y_+(t,y_0)= & \left(2-\sqrt{3}\right)^{-(1+\delta)t}\left(y_0+\left(\sqrt{3}\,y_0-3\right)t\right),\end{aligned}$$

and

$$\begin{aligned}x_-(t,y_0)= & e^{\varepsilon t}(\cosh t+y_0\sinh t),\\ y_-(t,y_0)= & e^{\varepsilon t}(y_0\cosh t+\sinh t).\end{aligned}$$

It is clear that the unstable separatrix of the saddle that intersects the discontinuous straight line $x=1$ does it at the point $(1,1)$, and the stable one does it at the point $(1,-1)$. Moreover the discontinuous system (3)–(4) has an sliding segment S with endpoints $(1,-\varepsilon)$ and $\left(1,\sqrt{3}+\log(2-\sqrt{3})(1+\delta)\right)$.

In what follows we assume that $\delta = 0$, until we do not say the contrary. Then an easy computation shows that

$$x_+(1,1) = 1, \quad \text{and} \quad y_+(1,1) = -1.$$

Therefore the saddle has a loop γ_ε, i.e. qualitatively we have again the Figure 1.

We claim the following two results:

(i) For $\varepsilon = 0$ all the orbits in the interior of the loop γ_ε (with the exception of the sliding segment S) have as α–limit the loop and its ω–limit is contained in the closure of the sliding segment S.
(ii) For $\varepsilon < 0$ sufficiently small one unstable limit cycle bifurcates from the loop γ_ε, and the loop persists.

Now we prove the claim. We start proving (i), so we assume that $\varepsilon = 0$. Let $y_0 \in (\sqrt{3} + \log(2-\sqrt{3}), 1)$. Then the equation $x_-(-t, y_0) = 1$ provides the time $t > 0$ which needs the orbit $(x_-(t, y_0), y_-(t, y_0))$ starting at the point $(1, y_0)$ when the time is zero for reaching in backward time the straight line $x = 1$. From $x_-(-t, y_0) = 1$ we get that

$$y_0 = (\cosh t - 1)\operatorname{csch} t. \tag{8}$$

Now we consider the equation $x_+(s, y_0) = 1$, it provides the time $s > 0$ which needs the orbit $(x_+(t, y_0), y_+(t, y_0))$ starting at the point $(1, y_0)$ when the time is zero for reaching in forward time the discontinuous straight line $x = 1$. From $x_+(s, y_0) = 1$ we obtain that

$$s = \frac{1}{\sqrt{3}-y_0} - \frac{1}{\log\left(2-\sqrt{3}\right)} W\left(\frac{\left(2-\sqrt{3}\right)^{1/(\sqrt{3}-y_0)} \log\left(2-\sqrt{3}\right)}{\sqrt{3}-y_0}\right). \tag{9}$$

Substituting in $y_-(-t, y_0) - y_+(s, y_0)$, first s given by (9) and after y_0 given by (8) we obtain, by Lemma 5, that $y_-(-t, y_0(t)) - y_+(s(y_0(t)), y_0(t)) = f(t) < 0$. So we have qualitatively the situation of the Figure 4.

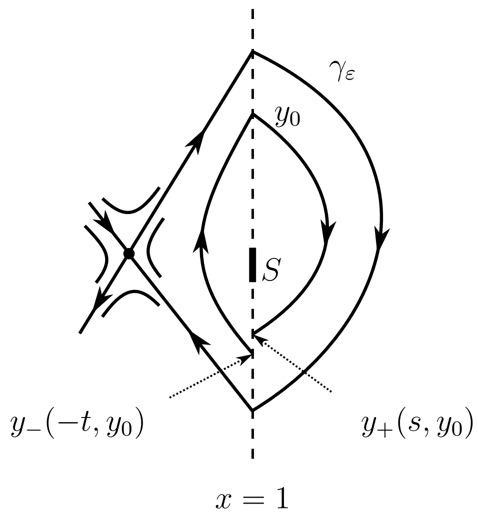

FIGURE 4. Showing that the $y_-(-t, y_0(t)) - y_+(s(y_0(t)), y_0(t)) = f(t) < 0$, for $t > 0$.

We have proved that the loop γ_ε exists for all ε. Therefore in its inner neighborhood we can define a Poincaré map P_ε. Using similar arguments to the proof of Theorem 1 for $\varepsilon = 0$ the domain of definition of this Poincaré map P_0 is contained in the interior I of the segment with endpoints $(1, \sqrt{3} + \log(2 - \sqrt{3}))$ and $(1, 1)$. Since $f(t) < 0$ we have that $P_0(y) > y$ for all $y \in I$ where P_0 is defined. This completes the proof of (i).

Again using the arguments of the proof of Theorem 1 we know that Theorem 4 also works for our discontinuous systems (3)–(4) having the loop γ_ε. Therefore, since the trace of our saddle is 2ε, Theorem 4 implies that for $\varepsilon < 0$ our loop γ_ε is locally stable, but for $\varepsilon = 0$ it was unstable, in fact all the orbits of the interior of the region bounded by the loop have their ω–limit contained in the closure of the sliding segment S. This forces the existence of an unstable limit cycle Γ_u for $\varepsilon < 0$ and sufficiently small which borns from the loop when $\varepsilon = 0$, and of course the loop remains as we have proved. This completes the proof of (ii).

Until here we have assumed that $\delta = 0$. Now if we take $\varepsilon \ll \delta < 0$ the unstable separatrix coming from the saddle until the point $(1, 1)$ does not connect with the stable separatrix at the point $(1, -1)$. We claim that this unstable separatrix intersects the straight line $x = 1$ nearby the point $(1, -1)$, but as close as this point as we want taking δ closer to zero. This shows that the unstable separatrix of the previous loop now goes in forward time towards that was the interior of the region bounded by the loop before this breaks, see Figure 5. Since inside the region bounded by the loop the qualitative behavior of the orbits persists if $\varepsilon \ll \delta < 0$, this implies the existence of a second stable limit cycle Γ_s which appears when the loop breaks at $\delta = 0$, and such that it contains in the region that it limits the limit cycle Γ_u. This completes the proof of Theorem 2, modulo the last claim that we shall prove now.

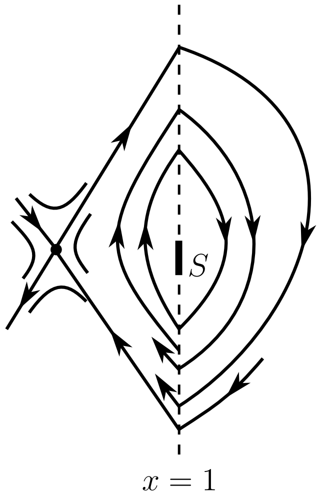

FIGURE 5. Creating the second limit cycle.

Let $\tau > 0$ be the time in order that the unstable manifold of the saddle starting at time zero at the point $(1, 1)$ reaches by first time the straight line $x = 1$. This τ must satisfy

$$x_+(\tau, 1) = \left(2 - \sqrt{3}\right)^{-\tau(\delta+1)} \left(\left(1 - \sqrt{3}\right)\tau + 1\right) = 1.$$

Solving this equation with respect to τ we obtain

$$\tau = \frac{1+\sqrt{3}}{2} - \frac{W\left(\frac{1}{2}\left(2-\sqrt{3}\right)^{\frac{1}{2}\left(1+\sqrt{3}\right)(\delta+1)}\left(1+\sqrt{3}\right)(\delta+1)\log\left(2-\sqrt{3}\right)\right)}{(\delta+1)\log\left(2-\sqrt{3}\right)}.$$

Then

$$y_+(\tau,1) = -1 + \frac{\log\left(2-\sqrt{3}\right)\left(\left(1+\sqrt{3}\right)\log\left(2-\sqrt{3}\right)-2w\right)}{2w(w+1)}\delta + O(\delta^2),$$

where

$$w = W\left(\frac{1}{2}\left(2-\sqrt{3}\right)^{\frac{1}{2}\left(1+\sqrt{3}\right)}\left(1+\sqrt{3}\right)\log\left(2-\sqrt{3}\right)\right).$$

Since $\varepsilon \ll \delta < 0$ and

$$\frac{\log\left(2-\sqrt{3}\right)\left(\left(1+\sqrt{3}\right)\log\left(2-\sqrt{3}\right)-2w\right)}{2w(w+1)} \approx -6.946475.$$

we get that $y_+(\tau,1) \gtrsim -1$. This proves the claim. □

A numerical computation of the initial point $(1, y_0)$ and the period of both limit cycles can be found in the Appendix.

4. Proof of Theorem 3

Since the arguments of the proof of Theorem 3 are essentially the same than the arguments used in the proof of Theorem 2 we only will present an sketch of the proof of Theorem 3.

Proof of Theorem 3. Since $0 < \delta \ll \varepsilon \ll 1$ clearly system (5) has a virtual stable node at the point $(-2,-2)$ with eigenvalues -2 and -1. On the other hand system (6) has a real saddle at the point $(-(3+\varepsilon-\delta)/(3+\varepsilon), 2\delta/(3+\varepsilon))$ with eigenvalues $-3+\varepsilon$ and $3+\varepsilon$.

The unstable separatrix of the saddle that intersects the discontinuous straight line $x = 0$ does it at the point $(0, 2)$, and the stable one does it at the point $(0, y_0) = (0, -(3+\varepsilon-3\delta)/(3+\varepsilon))$.

In what follows we assume that $\delta = 0$, until we do not say the contrary. Then the saddle has a loop γ_ε for all ε sufficiently small, i.e. qualitatively we have again the Figure 1. We have the following two results:

- (i) For $\varepsilon = 0$ all the orbits in the interior of the loop γ_ε (with the exception of the sliding segment S) have as ω–limit the loop and its α–limit is contained in the closure of the sliding segment S, see Figure 2.
- (ii) For $\varepsilon > 0$ sufficiently small one stable limit cycle Γ_s bifurcates from the loop γ_ε, and the loop persists.

Until here we have assumed that $\delta = 0$. Now if we take $0 < \delta \ll \varepsilon$ the stable node remains stable, and the unstable separatrix coming from the saddle until the point $(0, 2)$ does not connect with the stable at the point $(0, y_0)$. We claim this unstable separatrix intersect the straight line $x = 0$ a little under the point $(0, y_0)$, but as close as this point as we want taking δ closer to zero. This shows that the stable separatrix of the previous loop now goes in backward time towards that was the interior of the region bounded by the loop before this breaks, see Figure 3.

Since inside the region bounded by the loop the qualitative behavior of the orbits persists if $0 < \delta \ll \varepsilon$, the existence of a second (unstable) limit cycle Γ_u which appears when the loop breaks at $\delta = 0$, and such that it contains in the region that it limits the limit cycle Γ_s. This completes the main details of the proof of Theorem 3. □

5. Appendix. Numerical computations

Using the analytical expressions of the solutions of the discontinuous systems of Theorems 1, 2 and 3 we can compute numerically the limit cycles which born close to the loop. In Tables 2, 3 and 4 it can be seen how the parameter δ is much more small than ε. In each case, the computation of the flying times and the crossing points with the discontinuous line has been very accurate in order to distinguish both limit cycles. Because they appear very close to the homoclinic connection. When the corresponding system has a node to find the values of δ when ϵ decreases is much more difficult. Because the speed of the solutions of the vector field in this side is very high and these solutions remain a very short time in this side. It can be also seen how the period increase when the parameters tend to zero.

In Table 2 we provide the initial condition $(0, y_0)$ of the limit cycles provided by Theorem 1, its period T and the value of the initial condition $(0, 14/(5-3\varepsilon))$ of the homoclinic orbit of the loop for some values of ε and δ.

(ε, δ)	y_0	$14/(5-3\varepsilon)$	T
$(10^{-1}, 10^{-3})$	2.65075990890486	2.978723404255319	4.285823815839
$(10^{-1}, 10^{-3})$	2.97639485675640	2.978723404255319	6.410765616404
$(10^{-2}, 10^{-11})$	2.81690140519045	2.816901408450704	11.65332423071
$(10^{-2}, 10^{-11})$	2.81690140824725	2.816901408450704	12.76966772882

Table 2. Limit cycles bifurcating from the loop for systems (1)–(2).

In a similar way we can compute the limit cycles which born from the loop in Theorems 2 and 3. In Tables 3 and 4 we provide the initial condition $(1, y_0)$ and $(0, y_0)$, respectively, of these limit cycles and its period T for some values of ε and δ.

(ε, δ)	y_0	T
$(-10^{-1}, -10^{-6})$	$1 - 1.25407938437 \cdot 10^{-4}$	9.7966878121166
$(-10^{-1}, -10^{-6})$	$1 - 7.84031364774 \cdot 10^{-7}$	14.410868148689
$(-5 \cdot 10^{-2}, -10^{-10})$	$1 - 1.95383023905 \cdot 10^{-8}$	18.565748707201
$(-5 \cdot 10^{-2}, -10^{-10})$	$1 - 2.23289363663 \cdot 10^{-10}$	22.824475971018
$(-10^{-2}, -10^{-42})$	$1 - 8.08738858408 \cdot 10^{-39}$	88.528376162285
$(-10^{-2}, -10^{-42})$	$1 - 8.24684104247 \cdot 10^{-42}$	95.348406795279

Table 3. Limit cycles bifurcating from the loop for systems (3)–(4).

(ε, δ)	y_0	T
$(10^{-1}, 10^{-12})$	$2 - 5.53150104696 \cdot 10^{-9}$	7.62811921689722
$(10^{-1}, 10^{-12})$	$2 - 1.10946851355 \cdot 10^{-11}$	9.77010139695853
$(5 \cdot 10^{-2}, 10^{-22})$	$2 - 5.19255044848 \cdot 10^{-18}$	14.5568485263891
$(5 \cdot 10^{-2}, 10^{-22})$	$2 - 1.64587542565 \cdot 10^{-21}$	17.2879359348751
$(10^{-2}, 10^{-94})$	$2 - 2.88523068140 \cdot 10^{-90}$	70.0147745392481
$(10^{-2}, 10^{-94})$	$2 - 1.08334618077 \cdot 10^{-92}$	71.8825740291712

TABLE 4. Limit cycles bifurcating from the loop for systems (5)–(6).

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