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BIRTH OF LIMIT CYCLES FOR A CLASS OF CONTINUOUS AND DISCONTINUOUS DIFFERENTIAL SYSTEMS IN $(d+2)$–DIMENSION

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Abstract. The orbits of the reversible differential system $\dot{x} = -y$, $\dot{y} = x$, $\dot{z} = 0$, with $x, y \in \mathbb{R}$ and $z \in \mathbb{R}^d$, are periodic with the exception of the equilibrium points $(0, 0, z_1, \ldots, z_d)$. We compute the maximum number of limit cycles which bifurcate from the periodic orbits of the system $\dot{x} = -y$, $\dot{y} = x$, $\dot{z} = 0$, using the averaging theory of first order, when this system is perturbed, first inside the class of all polynomial differential systems of degree n, and second inside the class of all discontinuous piecewise polynomial differential systems of degree n with two pieces, one in $y > 0$ and the other in $y < 0$. In the first case this maximum number is $n^d(n-1)/2$, and in the second is $n^d(n-1)$.

1. Introduction and statements of the main results

Limit cycles have been used to model the behavior of many real process and different modern devices. In general to prove the existence of limit cycles is a very difficult problem. One way to produce limit cycles is perturbing differential systems that have a linear center. In this case, the limit cycles in a perturbed system bifurcate from the periodic orbits of the unperturbed center. The search for the maximum number of limit cycles that polynomial differential systems of a given degree can have is part of 16^{th} *Hilbert's Problem* and many contributions have been made in this direction, see for instance [11, 13, 16] and the references quoted therein.

Recently the theory of limit cycles has also been studied in discontinuous piecewise differential systems. The analysis of these systems can be traced from Andronov *et al.* [1] and still continues to receive attention by researchers. Discontinuous piecewise differential systems is a subject that have been developed very fast due to its strong applications to other branches of science. Currently such systems are one of the connections between mathematics, physics and engineering. These

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systems model several phenomena in control systems, impact in mechanical systems, nonlinear oscillations and economics see for instance [2, 3, 4, 6, 14, 28]. Recently they have been shown to be also relevant as idealized models for biology [15] and models of cell activity [7, 32, 33]. For more details see Teixeira [31] and all references therein.

As we have said it is not simple to determine the existence of limit cycles in a differential system. The simplest case for determining limit cycles is in planar continuous piecewise linear systems when they have only two linear differential systems separated by a straight line. Even in this simple case, only after a delicate analysis it was possible to show the existence of at most of one limit cycle for such systems, see [8] or an easier proof in [20].

Planar discontinuous piecewise linear differential systems with only two linearity regions separated by a straight line have been studied recently in [10, 12], among other papers. In [10] some results about the existence of two limit cycles appeared, so that the authors conjectured that the maximum number of limit cycles for this class of piecewise linear differential systems is exactly two. However in [12] strong numerical evidence about the existence of three limit cycles was obtained. As far we know the example in [12] represents the first discontinuous piecewise linear differential system with two zones with 3 limit cycles surrounding a unique equilibrium. Recently in [22] it is proved that such a system really has three limit cycles.

There are several papers studying the limit cycles of the continuous piecewise linear differential systems in $\mathbb{R}^3$, see for instance [5, 21, 23, 24, 25]. Our goal is study the periodic solutions of discontinuous piecewise polynomial differential systems in $\mathbb{R}^{d+2}$. More precisely the objective of this paper is to study the existence of limit cycles in continuous and discontinuous piecewise polynomial differential systems in $\mathbb{R}^{d+2}$, where the discontinuous differential system has two zones of continuity separated by a hyperplane. Without loss of generality we shall assume that the set of discontinuity is the hyperplane $y = 0$ in $\mathbb{R}^{d+2}$. So we consider the linear differential system in $\mathbb{R}^{d+2}$ given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = X(x, y, z),$$

which is reversible with respect to $\phi(x, y, z) = (x, -y, z)$, where the $\dot{}$ denotes derivative with respect to the time t and $x, y \in \mathbb{R}$, $z \in \mathbb{R}^d$. First we shall study the existence of limit cycles of the continuous polynomial differential system

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = X_\varepsilon(x, y, z), \tag{1}
$$

and after of the discontinuous piecewise polynomial differential system formed by two polynomial differential systems separated by the hyperplane $y = 0$, namely

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{cases} X_\varepsilon(x, y, z) & \text{if} \quad y > 0, \\ Y_\varepsilon(x, y, z) & \text{if} \quad y < 0, \end{cases} \tag{2}
$$

where

$$
\begin{aligned}
X_\varepsilon(x, y, z) &= X(x, y, z) + \varepsilon P(x, y, z), \\
Y_\varepsilon(x, y, z) &= X(x, y, z) + \varepsilon Q(x, y, z),
\end{aligned}
$$

with P and Q polynomials of degree n given by

$$
\begin{aligned}
P(x, y, z) &= (P(x, y, z, a), P(x, y, z, b), P(x, y, z, c_1), \ldots, P(x, y, z, c_d))^T, \\
Q(x, y, z) &= (Q(x, y, z, \alpha), Q(x, y, z, \beta), Q(x, y, z, \gamma_1), \ldots, Q(x, y, z, \gamma_l))^T,
\end{aligned}
$$

with

$$
P(x, y, z, a) = \sum_{i+j+k=0}^{n} a_{ijk} x^i y^j z^k, \quad Q(x, y, z, \alpha) = \sum_{i+j+k=0}^{n} \alpha_{ijk} x^i y^j z^k.
$$

In this work k is a multi-index and k denotes the expression $k_1 + \ldots + k_d$, z^k denotes the product $z_1^{k_1} \ldots z_d^{k_d}$ where $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$, and a_{ijk} denotes the coefficient $a_{ijk_1 \ldots k_d}$ of $x^i y^j z_1^{k_1} \ldots z_d^{k_d}$.

It is clear that systems (1) and (2) coincide for $\varepsilon = 0$ and they have linear centers at every plane $z =$ constant. In this paper we establish for $\varepsilon \neq 0$ sufficiently small the maximum number of limit cycle of these systems that bifurcate from the periodic orbits of these linear centers using the averaging theory of first order. The following result presents the results for the continuous case.

Theorem 1. *Using the averaging theory of first order for $|\varepsilon| \neq 0$ sufficiently small the maximum number of limit cycles of the polynomial differential system (1) is at most $n^d(n-1)/2$, and this number is reached.*

In the next theorem we present results for the discontinuous piecewise polynomial differential system (2).

Theorem 2. *Using the averaging theory of first order for $|\varepsilon| \neq 0$ sufficiently small the maximum number of limit cycles of the discontinuous* *piecewise polynomial differential system* (2) *is at most* $n^d(n-1)$*, and this number is reached.*

Corollary 3. *Under the assumptions of Theorem 1 if additionally* $a_{00k} = b_{00k} = 0$ *for all* k*, the limit cycles can be chosen as close to the origin of* $\mathbb{R}^{d+2}$ *as we want.*

Corollary 4. *Under the assumptions of Theorem 2 if additionally* $a_{00k} = b_{00k} = \alpha_{00k} = \beta_{00k} = 0$ *for all* k*, the limit cycles can be chosen as close to the origin of* $\mathbb{R}^{d+2}$ *as we want.*

Corollaries 3 and 4 provides information of the Hopf bifurcation of systems (1) and (2). More precisely, Corollaries 3 and 4 show that at least $n^d(n-1)/2$ and $n^d(n-1)$ limit cycles of systems (1) and (2) can bifurcate from the origin of $\mathbb{R}^{d+2}$, respectively. The results of Corollary Corollaries in the particular case $n = 2$ coincides with the result obtained in Theorem 1 of [26].

To prove these results we use the classical averaging theory, see for instance [30, 34] for a general introduction to this subject. This theory have been used for years to deal with continuous differential systems. The principle of averaging has been extended in many directions and recently in [19] the authors extend the averaging theory for detecting limit cycles of certain discontinuous piecewise differential systems, via the Brouwer degree and the regularization theory.

As far as we know this method is one of the best methods for determining limit cycles in discontinuous piecewise differential systems and has already been used by some authors. In [17] the method is used for determining the maximum number of limit cycles that bifurcate from the periodic solutions of some family of isochronous cubic polynomial centers perturbed by discontinuous piecewise cubic polynomial differential systems with two zones separated by a straight line. In [18] limit cycles for discontinuous piecewise quadratic differential systems with two zones was studied using the averaging theory. Also in [29] the averaging theory was applied to provide sufficient conditions for the existence of limit cycles of discontinuous perturbed planar centers when the discontinuity set is a union of regular curves.

We have organized this paper as follows. In section 2 we briefly present notation and basic concepts of the averaging theory of first order for continuous differential systems (see Theorem 5) and for discontinuous differential systems (see Theorem 6). In section 3 we present the proof of Theorem 1, and in section 4 we prove Theorem 2. Finally in section 5 we prove the Corollaries 3 and 4.

2. Basic results in averaging theory

In this section we present the basic results from the averaging theory of first order that we shall use for proving the results of this paper. The following theorem provides a method for studying the existence of periodic orbits of a differential system. For more details on the averaging method see for instance [34].

Let D be an open subset of $\mathbb{R}^n$. We denote the points of $\mathbb{R} \times D$ as (t, x), and we take the variable t as the time.

Theorem 5. *Consider the differential system*

$$\dot{x} = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \quad x(0) = 0, \tag{3}$$

where $F : \mathbb{R} \times D \to \mathbb{R}^n$ *and* $R : \mathbb{R} \times U \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ *are continuous functions,* $T-$*periodic in the first variable. Define the averaging function* $f : D \to \mathbb{R}^n$ *as*

$$f(x) = \int_0^T F(s, x)\ \mathrm{d}s, \tag{4}$$

and assume that

(i) *the functions* $F, R, D_x F, D_x^2 F$ *and* $D_x R$ *are defined, continuous and bounded by a constant* M *(independent of* ε*) in* $[0, \infty) \times D$ *and for* $\varepsilon \in (0, \varepsilon_0]$,

(ii) *for* $p \in D$ *with* $f(p) = 0$ *we have* $|J_f(p)| \neq 0$, *where* $|J_f(p)|$ *denotes the determinant of the Jacobian matrix of* f *evaluated at* p.

Then for $|\varepsilon| > 0$ *sufficiently small there exists a* $T-$*periodic solution* $x(t, \varepsilon)$ *of* (3) *such that* $x(0, \varepsilon) \to p$ *as* $\varepsilon \to 0$.

Now let $h : \mathbb{R} \times D \to \mathbb{R}$ be a $\mathcal{C}^1$ function with $0 \in \mathbb{R}$ as a regular value, and $\Sigma = h^{-1}(0)$. Let $X, Y : \mathbb{R} \times D \to \mathbb{R}^n$ be two continuous vector fields and assume that h, X and Y are $T-$periodic in the variable t. We define a *discontinuous piecewise differential system* as

$$\dot{x} = Z(t, x) = \begin{cases} X(t, x) & \text{if} \quad h(t, x) > 0, \\ Y(t, x) & \text{if} \quad h(t, x) < 0. \end{cases} \tag{5}$$

We rewrite the discontinuous differential system as follows. Consider the sign function defined in $\mathbb{R} \setminus \{0\}$ as

$$\operatorname{sign}(u) = \begin{cases} 1 & \text{if} \quad u > 0, \\ -1 & \text{if} \quad u < 0. \end{cases}$$

Then system (5) can be written as

$$\dot{x} = Z(t, x) = F_1(t, x) + \operatorname{sign}\left(h(t, x)\right) F_2(t, x),$$

where

$$F_1(t,x) = \frac{1}{2}\left(X(t,x) + Y(t,x)\right) \quad \text{and} \quad F_2(t,x) = \frac{1}{2}\left(X(t,x) - Y(t,x)\right).$$

The following theorem is a version of Theorem 5 for studying the periodic solutions of discontinuous differential systems.

Theorem 6. *Consider the discontinuous differential system*

$$\dot{x} = \varepsilon F(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{6}$$

with

$$\begin{aligned} F(t,x) &= F_1(t,x) + \operatorname{sign}\left(h(t,x)\right) F_2(t,x), \\ R(t,x,\varepsilon) &= R_1(t,x,\varepsilon) + \operatorname{sign}\left(h(t,x)\right) R_2(t,x,\varepsilon), \end{aligned}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ and $h : \mathbb{R} \times D \to \mathbb{R}$ are continuous functions, $T-$periodic in the variable t. We also suppose that h is a $\mathcal{C}^1$ function with 0 as a regular value and we denote $\Sigma = h^{-1}(0)$. Define the averaged function $f : D \to \mathbb{R}^n$ as

$$f(x) = \int_0^T F(s,x)\ \mathrm{d}s, \tag{7}$$

and assume that

- (i) *the functions F_1, F_2, R_1, R_2 and h are locally Lipschitz with respect to x;*
- (ii) $\dfrac{\partial h}{\partial t}(t,x) \neq 0$ *for all $(t,x) \in \Sigma$;*
- (iii) *for $p \in C$ with $f(p) = 0$, there exist a neighborhood $U \subset C$ of p such that $f(z) \neq 0$ for all $z \in \overline{U} \setminus \{p\}$ and $d_B(f, U, 0) \neq 0$ (d_B is the Brouwer degree of f in p).*

Then for $|\varepsilon| > 0$ sufficiently small there exists a $T-$periodic solutions $x(t,x)$ of system (6) such that $x(t,\varepsilon) \to p$ as $\varepsilon \to 0$.

For a proof of Theorem 6 see Theorem A and Proposition 2 in [19]. Here we emphasize that if f in (7) is C^1 then the hypotheses $d_B(f, U, 0) \neq 0$ holds if $|J_f(p)| \neq 0$, see for more details [27].

3. Proof of Theorem 1

Applying the change of variables

$$(x,y,z) = (r\cos\theta, r\sin\theta, z), \tag{8}$$

system (1) becomes

$$
\begin{aligned}
\dot{r} &= \varepsilon \sum_{i+j+k=0}^{n} r^{i+j} z^k \left(a_{ijk} \cos^{i+1}\theta \sin^j \theta + b_{ijk} \cos^i \theta \sin^{j+1}\theta \right), \\
\dot{\theta} &= 1 + \frac{\varepsilon}{r} \sum_{i+j+k=0}^{n} r^{i+j} z^k \big(a_{ijk} \cos^i \theta \sin^{j+1}\theta + \\
&\qquad\qquad\qquad b_{ijk} \cos^{i+1}\theta \sin^j \theta \big), \\
\dot{z}_l &= \varepsilon \sum_{i+j+k=0}^{n} c_{lijk}\, r^{i+j} z^k \cos^i \theta \sin^j \theta,
\end{aligned} \tag{9}
$$

for $l = 1, 2, \ldots, d$.

Essentially we study the limit cycles of system (9) with $\varepsilon \neq 0$ sufficiently small bifurcating from the periodic orbits of system (9) with $\varepsilon = 0$ contained in the cylindrical annulus

$$
\tilde{A} = \{(r, \theta, z) : r_0 \le r \le r_1, \theta \in \mathbb{S}^1, z \in \mathbb{R}^d\}. \tag{10}
$$

So for ε small enough $\dot{\theta} > 0$ for every $(r, z) \in \tilde{A}$.

Now taken as new independent variable θ instead of t so (1) in $\tilde{A}$ can be written as

$$
\begin{pmatrix} r' \\ z_1' \\ \vdots \\ z_d' \end{pmatrix} = \varepsilon \begin{pmatrix} F_1(\theta, r, z) \\ F_2(\theta, r, z) \\ \vdots \\ F_{d+1}(\theta, r, z) \end{pmatrix} + \mathcal{O}(\varepsilon^2), \tag{11}
$$

where the $'$ denotes derivative with respect to the variable θ, and

$$
\begin{aligned}
F_1(\theta, r, z) &= \sum_{i+j+k=0}^{n} r^{i+j} z^k \left(a_{ijk} \cos^{i+1}\theta \sin^j \theta + b_{ijk} \cos^i \theta \sin^{j+1}\theta \right), \\
F_{l+1}(\theta, r, z) &= \sum_{i+j+k=0}^{n} r^{i+j} z^k \; c_{lijk} \cos^i \theta \sin^j \theta,
\end{aligned} \tag{12}
$$

for $l = 1, 2, \ldots, d$.

To apply Theorem 5 to system (11), we compute the averaged function $f = (f_1, f_2, \ldots, f_{d+1})$ given in (4) with $T = 2\pi$ and we obtain

$$
\begin{aligned}
f_1(r,z) &= \int_0^{2\pi} F_1(s,r,z)\ \mathrm{d}s \\
&= \sum_{\substack{i+j+k=0 \\ i\ odd,\ j\ even}}^{n} r^{i+j} z^k\ a_{ijk} \int_0^{2\pi} \cos^{i+1}\theta \sin^j\theta\ \mathrm{d}\theta \\
&\quad + \sum_{\substack{i+j+k=0 \\ i\ even,\ j\ odd}}^{n} r^{i+j} z^k\ b_{ijk} \int_0^{2\pi} \cos^{i}\theta \sin^{j+1}\theta\ \mathrm{d}\theta, \\
f_{l+1}(r,z) &= \int_0^{2\pi} F_{l+1}(s,r,z)\ \mathrm{d}s \\
&= \sum_{\substack{i+j+k=0 \\ i,j\ even}}^{n} r^{i+j} z^k\ c_{lijk} \int_0^{2\pi} \cos^{i}\theta \sin^{j}\theta\ \mathrm{d}\theta, \qquad (13)
\end{aligned}
$$

for $l = 1, 2, \ldots, d$.

We split the proof in two parts. First assume n is odd, so

$$f_1(r,z) = A_1 r + A_3 r^3 + \ldots + A_n r^n$$

where

$$
\begin{aligned}
A_p = &\sum_{k=0}^{n-p} \sum_{\substack{i+j=p \\ i\ odd, j\ even}} z^k\ a_{ijk} \int_0^{2\pi} \cos^{i+1}\theta \sin^j\theta\ \mathrm{d}\theta \\
&+ \sum_{k=0}^{n-p} \sum_{\substack{i+j=p \\ i\ even, j\ odd}} z^k\ b_{ijk} \int_0^{2\pi} \cos^{i}\theta \sin^{j+1}\theta\ \mathrm{d}\theta. \qquad (14)
\end{aligned}
$$

We write $f_1 = r\bar{f}_1$ with

$$\bar{f}_1(r,z) = A_1 + A_3 r^2 + \ldots + A_n r^{n-1}.$$

Since $r > 0$ it is sufficient to solve $(\bar{f}_1, f_2, \ldots, f_{d+1}) = (0, \ldots, 0)$ to determine the number of solutions of $f \equiv 0$. As $\bar{f}_1$ is a polynomial in the variables r and $z \in \mathbb{R}^d$ of degree $n-1$ and f_{l+1} are polynomials in the variables r and $z \in \mathbb{R}^d$ of degree n for $l = 1, 2, \ldots, d$, by Bézout's theorem (see [9]) $(\bar{f}_1, f_2, \ldots, f_{d+1})$ has at most $n^d(n-1)$ solutions. However $\bar{f}_1$ is even on variable r then we consider only solutions with

$r > 0$, in this case the maximum number of solutions of $f \equiv 0$ is $n^d(n-1)/2$.

Now we prove that this number is reached. For this, we exhibit a particular case for which this occurs. Let $a_{ij0}, b_{ij0} \neq 0$ and we take zero all the other a_{ijk}, b_{ijk}, then $\bar{f}_1(r,z)$ is a real polynomial that does not depend of $z \in \mathbb{R}^d$ of degree $n-1$, that is

$$\bar{f}_1 = A_1 + A_3 r^2 + \ldots + A_n r^{n-1}$$

where A_p is

$$\sum_{\substack{i+j=p \\ i \text{ odd} \\ j \text{ even}}} a_{ij0} \int_0^{2\pi} \cos^{i+1}\theta \sin^j \theta \, \mathrm{d}\theta + \sum_{\substack{i+j=p \\ i \text{ even} \\ j \text{ odd}}} b_{ij0} \int_0^{2\pi} \cos^{i}\theta \sin^{j+1} \theta \, \mathrm{d}\theta.$$

On the other hand, we take $c_{lijk} = 0$ if $i, j, k_1, k_{l-1}, k_{l+1}, \ldots, k_d \neq 0$ for each $l = 1, 2, \ldots, d$ so that

$$f_{l+1}(r,z) = \sum_{k_l=0}^{n} {z_l}^{k_l} \; (2\pi \; c_{l00k_l}).$$

In this particular case, we can take a_{ij0}, b_{ij0} and c_{l00k_l} in order that all coefficients of $\bar{f}_1$ and f_{l+1} are linearly independent for all $l = 1, 2, \ldots, d$. So we can choose these coefficients in such a way that $\bar{f}_1$ has $(n-1)/2$ simple positive real roots and f_{l+1} has n simple real roots for each $l = 1, 2, \ldots, d$. Then $(\bar{f}_1, f_2, \ldots, f_{d+1})$ has $n^d(n-1)/2$ solutions with $r > 0$.

Now assume n is even. Then $f = r\bar{f}_1$ where

$$\bar{f}_1(r,z) = A_1 + A_3 r^2 + \ldots + A_{n-1} r^{n-2}.$$

with A_p given in (14).

Note that $\bar{f}_1$ has degree $n-1$ as a polynomial in the variables r and z and f_{l+1} given in (13) has degree n as polynomials in the variables r and z for all $l = 1, 2, \ldots, d$, so $(\bar{f}_1, f_2, \ldots, f_{d+1})$ have at most $n^d(n-1)/2$ solutions of type (r,z) with $r > 0$.

To prove that this number is reached we consider $a_{10k_1}, b_{01k_1} \neq 0$ and we take zero all the other a_{ijk}, b_{ijk} then

$$\bar{f}_1(r,z) = A_1 = \sum_{k_1=0}^{n-1} z_1^{k_1} \left(a_{10k_1} \int_0^{2\pi} \cos^2\theta \, \mathrm{d}\theta + b_{01k_1} \int_0^{2\pi} \sin^2\theta \, \mathrm{d}\theta \right)$$

is a complete polynomial on variable z_1 of degree $n-1$. Now we take $c_{1ij0}, c_{l00k_l} \neq 0$ for each $l = 2, 3 \ldots, d$ and we take zero all the other c_{lijk}

so that

$$f_2(r,z) = \sum_{\substack{i+j=0 \\ i,j\ even}}^{n} r^{i+j}\left(c_{1ij0}\int_0^{2\pi} \cos^i\theta \sin^j\theta \ \mathrm{d}\theta\right),$$

$$f_{l+1}(r,z) = \sum_{k_l=0}^{n} z_l^{k_l}\ (2\pi\ c_{l00k_l}).$$

Here we take a_{10k_1}, b_{01k_1}, c_{1ij0} and c_{l00k_l} in such a way for that all coefficients of $\bar{f}_1$ and f_{l+1} for $l = 1, 2, \ldots, d$, are linearly independent. Therefore we can choose these coefficients in order that $\bar{f}_1$ has $n-1$ simple real roots, f_2 has $n/2$ simple positive real roots and f_{l+1} has n simple real roots for each $l = 2, 3, \ldots, d$. In this case $(\bar{f}_1, f_2, \ldots, f_{d+1}) = (0, \ldots, 0)$ has $n^d(n-1)/2$ solutions with $r > 0$. Furthermore by independence of the coefficients these solutions can be taken in a way that the Jacobian of f in all these solutions is nonzero.

This completes the proof of Theorem 1.

4. Proof of Theorem 2

We apply again the change of coordinates given in (8) for $Y_\varepsilon(x, y, z)$. For $X_\varepsilon(x, y, z)$ we use the calculations done in the previous section. Analogously to what we did in the previous section we shall study the periodic solutions of system (2) with $\varepsilon = 0$ contained in the cylindrical annulus (10) which can be prolonged to limit cycles of system (2) with $\varepsilon \neq 0$ sufficiently small. In $\tilde{A}$ we have for ε small enough $\dot{\theta} > 0$ for all $(r, \theta, z) \in \tilde{A}$.

Taking θ as independent variable system (2) in $\tilde{A}$ becomes

$$\begin{pmatrix} r' \\ z_1' \\ \vdots \\ z_d' \end{pmatrix} = \begin{cases} \varepsilon \begin{pmatrix} F_1(\theta,r,z) \\ F_2(\theta,r,z) \\ \vdots \\ F_{d+1}(\theta,r,z) \end{pmatrix} + \mathcal{O}(\varepsilon^2) \quad \text{if} \quad h(\theta,r,z) > 0, \\ \\ \varepsilon \begin{pmatrix} G_1(\theta,r,z) \\ G_2(\theta,r,z) \\ \vdots \\ G_{d+1}(\theta,r,z) \end{pmatrix} + \mathcal{O}(\varepsilon^2) \quad \text{if} \quad h(\theta,r,z) < 0, \end{cases} \tag{15}$$

with

$$G_1(\theta, r, z) = \sum_{i+j+k=0}^{n} r^{i+j} z^k \left(\alpha_{ijk} \cos^{i+1}\theta \sin^j\theta + \beta_{ijk}\cos^i\theta \sin^{j+1}\theta\right),$$
$$G_{l+1}(\theta, r, z) = \sum_{i+j+k=0}^{n} r^{i+j} z^k \gamma_{lijk} \cos^i\theta \sin^j\theta,$$

for $l = 1, 2, \ldots, d$, with F_1 and F_2 given in (12) and $h(\theta, r, z) = \sin\theta$. So for $(\theta, r, z) \in h^{-1}(0)$ we have $\dfrac{\partial h}{\partial \theta}(\theta, r, z) \mid_{\theta\in\{0,\pi\}} = \cos\theta \mid_{\theta\in\{0,\pi\}} = \pm 1 \neq 0$.

We must study the zeros of the averaged function f given in (7), namely $f = (f_1, f_2, \ldots, f_{d+1})$ where

$$
\begin{aligned}
f_1(r, z) &= \int_0^{\pi} F_1(s, r, z)\ \mathrm{d}s + \int_{\pi}^{2\pi} G_1(s, r, z)\ \mathrm{d}s \\
&= \sum_{\substack{i+j+k=0 \\ i\ odd}}^{n} r^{i+j} z^k \left(a_{ijk} + (-1)^j \alpha_{ijk}\right) \int_0^{\pi} \cos^{i+1}\theta \sin^j\theta\ \mathrm{d}\theta \\
&\quad + \sum_{\substack{i+j+k=0 \\ i\ even}}^{n} r^{i+j} z^k \left(b_{ijk} - (-1)^j \beta_{ijk}\right) \int_0^{\pi} \cos^i\theta \sin^{j+1}\theta\ \mathrm{d}\theta, \\
f_{l+1}(r, z) &= \int_0^{\pi} F_{l+1}(s, r, z)\ \mathrm{d}s + \int_{\pi}^{2\pi} G_{l+1}(s, r, z)\ \mathrm{d}s \\
&= \sum_{\substack{i+j+k=0 \\ i\ even}}^{n} \left(c_{lijk} + (-1)^j \gamma_{lijk}\right) r^{i+j} z^k \int_0^{\pi} \cos^i\theta \sin^j\theta\ \mathrm{d}\theta,
\end{aligned}
$$

for $l = 1, 2, \ldots, d$.
Therefore we can to write $f_1(r, z)$ as

$$A_1 r + A_2 r^2 + \ldots + A_n r^n \tag{16}$$

with

$$A_p = \sum_{k=0}^{n-p} \sum_{\substack{i+j=p \\ i \text{ odd}}} z^k \left(a_{ijk} + (-1)^j \alpha_{ijk}\right) \int_0^\pi \cos^{i+1}\theta \sin^j \theta \, \mathrm{d}\theta$$

$$+ \sum_{k=0}^{n-p} \sum_{\substack{i+j=p \\ i \text{ even}}} z^k \left(b_{ijk} - (-1)^j \beta_{ijk}\right) \int_0^\pi \cos^{i}\theta \sin^{j+1} \theta \, \mathrm{d}\theta,$$

for $l = 1, 2, \ldots, d$.

Note that $f_1 = r\bar{f}_1$ with $\bar{f}_1(r, z) = A_1 + A_2 r + \ldots + A_n r^{n-1}$. As $r > 0$, to know the solutions of $(f_1, f_2, \ldots, f_{d+1}) = (0, \ldots, 0)$ is equivalent to solve $(\bar{f}_1, f_2, \ldots, f_{d+1}) = (0, \ldots, 0)$. But, $\bar{f}_1$ is a polynomial on variables r and z of degree $n-1$ and f_{l+1} are polynomials on variables r and z of degree n for each $l = 1, 2, \ldots, d$. By Bézout's theorem f has at most $n^d(n-1)$ solutions (with $r > 0$).

To prove that this number is reached we choose a particular example. So let $a_{ij0}, \alpha_{ij0}, b_{ij0}, \beta_{ij0} \neq 0$ and we take zero all the other $a_{ijk}, \alpha_{ijk}, b_{ijk}, \beta_{ijk}$, then $\bar{f}_1$ is a real polynomial of degree $n-1$ that does not depend of z, that is $\bar{f}_1 = A_1 + A_2 r + \ldots + A_n r^{n-1}$ with

$$A_p = \sum_{\substack{i+j=p \\ i \text{ odd}}} \left(a_{ij0} + (-1)^j \alpha_{ij0}\right) \int_0^\pi \cos^{i+1}\theta \sin^j \theta \, \mathrm{d}\theta$$

$$+ \sum_{\substack{i+j=p \\ i \text{ even}}} \left(b_{ij0} - (-1)^j \beta_{ij0}\right) \int_0^\pi \cos^{i}\theta \sin^{j+1} \theta \, \mathrm{d}\theta,$$

for $1 \leq p \leq n-1$.

Analogously for each $l = 1, 2, \ldots, d$ we take $c_{lijk} = \gamma_{lijk} = 0$ if $i, j, k_1, \ldots, k_{l-1}, k_{l+1}, \ldots, k_d \neq 0$ so that

$$f_{l+1}(r, z) = \sum_{k_l=0}^{n} z_l^{k_l} \, (c_{l00k} + \gamma_{l00k})\pi$$

do not depends of r.

Under such conditions all coefficients of $\bar{f}_1$ and f_{l+1} for $l = 1, 2, \ldots, d$ can be taken to be linearly independent from the appropriate choice of a_{ij0}, α_{ij0}, b_{ij0}, β_{ij0}, c_{l00k} and γ_{l00k}. Such values can be taken so that $\bar{f}_1$ is a complete real polynomial on variable r of degree $n-1$, and therefore it can have $n-1$ simple positive real roots, and each

f_{l+1} is a complete real polynomial on variable z_l with n simple real roots for $l = 1, 2, \ldots, d$. So for this particular case the number of solutions of $(\bar{f}_1, f_2, \ldots, f_{l+1}) = (0, \ldots, 0)$ is $n^d(n-1)$. By independence of the coefficients such solutions can be taken so that the Jacobian of f evaluated at them is nonzero.

By Theorem 6 this completes the proof of Theorem 2.

5. Proofs of Corollaries 3 and 4

Proof of Corollary 3. To prove Corollary 3 we follow the steps of the proof presented in section 3, however we highlight some differences. We consider the change of coordinates given in (8) applied to system (1) taking $a_{00k} = b_{00k} = 0$ for all k and instead of (9) we obtain

$$\dot{r} = \varepsilon \sum_{i+j+k=0}^{n} r^{i+j} z^k \left(a_{ijk} \cos^{i+1}\theta \sin^j\theta + b_{ijk} \cos^i\theta \sin^{j+1}\theta \right),$$

$$\dot{\theta} = 1 + \varepsilon \sum_{i+j+k=0}^{n} r^{i+j} z^k \left(a_{ijk} \cos^{i}\theta \sin^{j+1}\theta + b_{ijk} \cos^{i+1}\theta \sin^{j}\theta \right),$$

$$\dot{z}_l = \varepsilon \sum_{i+j+k=0}^{n} c_{lijk}\ r^{i+j} z^k \cos^i\theta \sin^j\theta,$$

for $l = 1, 2, \ldots, d$.

As r does not appear in the denominator of $\dot{\theta}$, if ε is sufficiently small $\dot{\theta} > 0$ for every (r, θ, z) in a ball B of an arbitrary given radius around the origin of $\mathbb{R}^{d+2}$. Now in the ball B r can be approximated to the zero as we want, this cannot occur working with the cylindrical annulus $\tilde{A}$ of section 3. From here the calculations are done analogously to section 3, and we obtain the same maximum number of zeros of the averaging function for the new system (11) with $a_{00k} = b_{00k} = 0$. □

Proof of Corollary 4. The same above argument can be used for proving Corollary 4 and we follow the steps of the proof presented in section 4. More precisely, we apply the change of coordinates (8) to system (2) considering $a_{00k} = \alpha_{00k} = b_{00k} = \beta_{00k} = 0$ for all k in order to obtain an expression for $\dot{\theta}$ in both systems $y > 0$ and $y < 0$ with denominator that does not depend on r. So for ε sufficiently small $\dot{\theta} > 0$ for every (r, θ, z) in the ball B of the proof of Corollary 3. The rest of the proof is done as in section 4 for obtaining the maximum number of zeros of the averaging function for the new system (15) with $a_{00k} = \alpha_{00k} = b_{00k} = \beta_{00k} = 0$. Again in the ball B r can be approximated to zero as we want. □

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