INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 2681–2686 doi:10.1088/0305-4470/38/12/010

Formal and analytic integrability of the Lorenz system

Jaume Llibre[1] **and Clàudia Valls**[2]

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
[2] Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

E-mail: jllibre@mat.uab.es and cvalls@math.ist.utl.pt

Received 24 September 2004, in final form 18 January 2005
Published 9 March 2005
Online at stacks.iop.org/JPhysA/38/2681

Abstract
The well-known Lorenz system can be written as $\dot{x} = s(y-x)$, $\dot{y} = rx-y-xz$ and $\dot{z} = -bz + xy$. Here, we study the first integrals of the Lorenz system that can be described by formal power series. In particular, if $s \neq 0$ and, either b is not a negative rational number, or b is a negative rational number and $k_1 b + k_2(1+s) \neq 0$, for all k_1 and k_2 non-negative integers with $k_1 + k_2 > 0$, then the Lorenz system has no analytic first integrals in a neighbourhood of the origin.

PACS numbers: 04.25.Nx, 02.30.Hq, 02.40.−k, 02.40.Ma
Mathematics Subject Classification: 34C05, 34A34, 34C14

1. Introduction

The Lorenz system (see [9]):

$$\dot{x} = s(y-x), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = -bz + xy, \tag{1}$$

is a famous dynamical model (see for instance [10]), where x, y and z are real variables; and s, r and b are real parameters. This system has been intensively investigated as a dynamical system (see for instance [14]), mainly for studying its strange attractors, the more classical one appears for the parameter values $s = 10$, $b = 8/3$ and $r = 28$. From the point of view of integrability it was also intensively studied using different integrability theories (for example, see [1, 3–7, 12, 13, 15–18]). But in this paper we are interested in its formal power series first integrals and in its analytical first integrals.

The associated vector field of the Lorenz system is

$$X = s(y-x)\frac{\partial}{\partial x} + (rx - y - xz)\frac{\partial}{\partial y} - (bz - xy)\frac{\partial}{\partial z}. \tag{2}$$

A *first integral* of system (1) is a non-constant function $H = H(x, y, z)$ satisfying

$$XH = s(y-x)\frac{\partial H}{\partial x} + (rx - y - xz)\frac{\partial H}{\partial y} - (bz - xy)\frac{\partial H}{\partial z} = 0.$$