Global analytic first integrals for the real planar Lotka-Volterra system

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(Received 4 December 2006; accepted 5 February 2007; published online 21 March 2007)

We provide the complete classification of all Lotka-Volterra systems of the form $\dot{x}=x(ax+by+c)$ and $\dot{y}=y(Ax+By+C)$ in $\mathbb{R}^2$ having a global analytic first integral. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2713076]

I. INTRODUCTION

The nonlinear ordinary differential equations appear in a natural way in many branches of applied mathematics, physics, chemistry, economy, etc. For a two-dimensional system, the existence of a first integral determines completely its phase portrait. For such systems the notion of integrability is based on the existence of a first integral. Then a natural question is: *Given a system of ordinary differential equations in $\mathbb{R}^2$ depending on parameters, how do we recognize the values of the parameters for which the system has a first integral*?

The easiest planar integrable systems are the Hamiltonian ones; i.e., the systems in $\mathbb{R}^2$ that can be written as

$$\dot{x} = -\frac{\partial H}{\partial y}, \quad \dot{y} = \frac{\partial H}{\partial x},$$

for some function $H:\mathbb{R}^2 \to \mathbb{R}$ of class C^2. The planar integrable systems which are not Hamiltonian are in general very difficult to detect. The goal of this paper is to present the complete classification of the global analytic first integrals for the quadratic Lotka-Volterra systems,

$$\dot{x} = x(ax + by + c),$$

$$\dot{y} = y(Ax + By + C), \tag{1}$$

in $\mathbb{R}^2$. Here *a global analytic first integral* or simply *an analytic first integral* is a nonconstant analytic function $H:\mathbb{R}^2 \to \mathbb{R}$, whose domain of definition is the whole $\mathbb{R}^2$, and it is constant on the solutions of system (1). This last assertion means that for any solution $(x(t),y(t))$ of Eq. (1), we have

$$\frac{dH}{dt}(x(t),y(t)) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0.$$

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