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Darboux integrability and algebraic invariant surfaces for the Rikitake system

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In this paper, we study the Darboux integrability of the Rikitake system $\dot{x}=-\mu x+yz$, $\dot{y}=-\mu y+x(z-a)$, $\dot{z}=1-xy$. More precisely, we characterize all the invariant algebraic surfaces, the exponential factors, and the polynomial, rational, and Darboux first integrals of this system according to the values of its parameters a and μ. *© 2008 American Institute of Physics.* [DOI: 10.1063/1.2897983]

I. INTRODUCTION

A simple mechanical model used to study the reversals of the earth's magnetic field is a two-disk dynamo system idealized by the Japanese geophysicist Rikitake[15] and known now as the Rikitake system. The model consists of two identical single Faraday-disk dynamos of Bullard type coupled together, whose dynamics is governed by the following three dimensional system of nonlinear differential equations:

$$\dot{x}=-\mu x+yz, \quad \dot{y}=-\mu y+x(z-a), \quad \dot{z}=1-xy. \tag{1}$$

where a and μ are real parameters and the dot indicates derivative with respect to the time variable t. For a physical meaning of the variables x, y, and z, see Refs. 8 and 14, or 15. Besides its physical interest, the Rikitake system has been widely investigated from the dynamical point of view due to the richness of the behavior presented by its solutions, see, for instance, Refs. 8, 10, and 13–15 and the references therein. It has also been considered from the points of view of integrability and stochasticity.

With respect to integrability, there have been several approaches. In particular, in Ref. 17 the authors used the Painlevé method to gain some information about the possibility of existence of first integrals and thus find integrability regions. Furthermore, complex analysis techniques are used in Ref. 9 to obtain similar results, while in Refs. 6, 11, and 13 the authors use geometrical tools and algebraic methods to find invariant algebraic surfaces of system (1). On the other hand, some numerical work exists that seems to indicate for convenient values of the parameters some Lorenz-like chaotic behavior of this system, see Ref. 10 and also Refs. 1, 8, 14, and 16 for other investigations concerning dynamical aspects of the system.

We recall that a *first integral* is a nonconstant function $H:\mathbb{R}^3\to\mathbb{R}$ that is constant on all solution curves $(x(t),y(t),z(t))$ of the system, that is, $H(x(t),y(t),z(t))=\text{const}$ for all values of t for which the solution $(x(t),y(t),z(t))$ is defined on $\mathbb{R}^3$. We say that system (1) is (completely) *integrable* if it has two linearly independent first integrals.

Theorem 1: *If $a=\mu=0$, then the Rikitake system (1) is (completely) integrable with first integrals*

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