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ON THE LOCAL ANALYTIC INTEGRABILITY AT THE SINGULAR POINT OF A CLASS OF LIÉNARD ANALYTIC DIFFERENTIAL SYSTEMS

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ABSTRACT. We consider the Liénard analytic differential systems $\dot{x} = y$, $\dot{y} = -cx - f(x)y$, with $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ an analytic function. Then for such systems we characterize the existence of local analytic first integrals in a neighborhood of the singular point located at the origin.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

One of the more classical problems in the qualitative theory of planar analytic differential systems in $\mathbb{R}^2$ is to characterize the existence of analytic first integrals in a neighborhood of an isolated singular point.

One of the best and oldest results in this direction is the analytic nondegenerate center theorem. In order to be more precise we recall some definitions. A singular point is a *nondegenerate center* if it is a center with eigenvalues purely imaginary. If a real planar analytic system has a nondegenerate center, then after an affine change of variables and a rescaling of the independent variable t, it can be written in the form

$$\begin{aligned} \dot{x} &= y + X(x,y), \\ \dot{y} &= -x + Y(x,y), \end{aligned} \tag{1}$$

where $X(x,y)$ and $Y(x,y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin. The dot will always denote derivative with respect to the independent variable t.

Let U be an open subset of $\mathbb{R}^2$, $H : U \to \mathbb{R}$ be a nonconstant analytic function and $\mathcal{X}$ be an analytic vector field defined on U. Then H is an *analytic first integral of* $\mathcal{X}$ *in* U if H is constant on the solutions of $\mathcal{X}$, i.e. if $\mathcal{X}H|_U = 0$. If $U = \mathbb{R}^2$, then H is called a *global analytic first integral of* $\mathcal{X}$.

The next result is due to Poincaré [7], [8] and Liapunov [5]; see also Moussu [6].

Theorem 1 (Analytic nondegenerate center theorem). *The analytic differential system* (1) *has a nondegenerate center at the origin if and only if there exists an analytic first integral defined in a neighborhood of the origin.*

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