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Liouvillian integrability of the FitzHugh–Nagumo systems

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ABSTRACT

We study the Liouvillian integrability of the FitzHugh–Nagumo systems $\dot{x} = x(1-x)(x+a) - y + b$, $\dot{y} = d(x - cy)$, in $\mathbb{R}^2$ with parameters $a, b, c, d \in \mathbb{R}$.

1. Introduction

In this paper we study the Liouvillian integrability of the FitzHugh–Nagumo systems. These equations were introduced in papers of FitzHugh [1] and Nagumo et al. [2] as one of the simplest models describing the excitation of neural membranes and the propagation of nerve impulses along an axon. Looking at MathSciNet you can find several hundred papers published on these systems or related with them. For an introduction to the Liouvillian integrability theory of the polynomial differential equations see for instance [3–7].

We explicitly consider the following FitzHugh–Nagumo equation:

$$\begin{aligned} \dot{x} &= x(1-x)(x+a) - y + b, \\ \dot{y} &= d(x - cy), \end{aligned} \tag{1}$$

in $\mathbb{R}^2$ with parameters $a, b, c, d \in \mathbb{R}$. These differential systems (or ones with slight differences) appear in many papers; see for instance [8,9] and Chapter 14 of [10]. But as far as we know all the papers working with these systems have been interested in their dynamics, not in their integrability (see for example [11]).

The main objective of this paper is to study the Liouvillian integrability of the FitzHugh–Nagumo system (1) in $\mathbb{R}^2$ depending on the parameters $a, b, c, d \in \mathbb{R}$.

Let $U \subset \mathbb{R}^2$ be an open subset. We say that the non-constant function $H: U \to \mathbb{R}$ is a *first integral* of the polynomial vector field

$$\mathcal{X} = (x(1-x)(x+a) - y + b)\frac{\partial}{\partial x} + d(x - cy)\frac{\partial}{\partial y} \tag{2}$$

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