PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 9, September 2010, Pages 3229–3239
S 0002-9939(10)10338-4
Article electronically published on April 9, 2010

LIOUVILLIAN FIRST INTEGRALS FOR LIÉNARD POLYNOMIAL DIFFERENTIAL SYSTEMS

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(Communicated by Yingfei Yi)

ABSTRACT. We characterize the Liouvillian first integrals for the Liénard polynomial differential systems of the form $x' = y$, $y' = -cx - f(x)y$, with $c \in \mathbb{R}$ and $f(x)$ is an arbitrary polynomial. For obtaining this result we need to find all the Darboux polynomials and the exponential factors of these systems.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

One of the more classical problems in the qualitative theory of planar differential systems depending on parameters is to characterize the existence or not of first integrals.

We consider the system

$$x' = y, \quad y' = -cx - f(x)y, \tag{1}$$

which we call the *generalized classical Liénard differential system*, where x and y are complex variables and the prime denotes derivative with respect to the time t, which can be either real or complex. Such differential systems appear in several branches of the sciences, such as biology, chemistry, mechanics, electronics, etc. For $c = 1$ the Liénard differential systems (1) are called the *classical Liénard systems*.

The main objective of this paper is to study the *Liouvillian first integrals* of systems (1) depending on the polynomial function $f(x)$ and on $c \in \mathbb{R}$.

Let $U \subset \mathbb{C}^2$ be an open set. We say that the nonconstant function $H: \mathbb{C}^2 \to \mathbb{C}$ is a first integral of the polynomial vector field X on U if $H(x(t), y(t)) =$ constant for all values of t for which the solution $(x(t), y(t))$ of X is defined on U. Clearly H is a first integral of X on U if and only if $XH = 0$ on U.

A *Liouvillian first integral* is a first integral H which is a Liouvillian function, that is, roughly speaking, which can be obtained "by quadratures" of elementary functions. For a precise definition, see [13]. The study of the Liouvillian first integrals is a classical problem of the integrability theory of the differential equations which goes back to Liouville; see for details again [13].

As far as we know, the Liouvillian first integral of some multi-parameter family of planar polynomial differential systems has only been classified for the Lotka-Volterra system; see [1, 5, 8, 9, 10, 11].

Received by the editors September 10, 2009 and, in revised form, December 16, 2009.
2010 *Mathematics Subject Classification.* Primary 37K10, 34D30.
Key words and phrases. Darboux polynomials, exponential factors, Liouvillian first integrals, Liénard polynomial differential systems.