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Analytic integrability of quadratic–linear polynomial differential systems

JAUME LLIBRE† and CLÀUDIA VALLS‡

† *Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain*
(e-mail: jllibre@mat.uab.cat)

‡ *Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001, Lisboa, Portugal*
(e-mail: cvalls@math.ist.utl.pt)

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Abstract. For the quadratic–linear polynomial differential systems with a finite singular point, we classify the ones which have a global analytic first integral, and provide the explicit expression of their first integrals.

1. *Introduction*

For a two-dimensional system the existence of a first integral determines completely its phase portrait. For such systems the notion of integrability is based on the existence of a first integral. Then a natural question arises: *given a system of ordinary differential equations in* $\mathbb{R}^2$ *depending on parameters, how to recognize the values of such parameters for which the system has a first integral?*

The planar integrable systems which are not Hamiltonian, i.e. the systems in $\mathbb{R}^2$ that cannot be written as $x' = -\partial H/\partial y$, $y' = \partial H/\partial x$ for some function $H : \mathbb{R}^2 \to \mathbb{R}$ of class C^2, are in general very difficult to detect.

Let P and Q be two real polynomials in the variables x and y, then we say that the system

$$x' = P(x, y), \quad y' = Q(x, y),$$

is a *quadratic polynomial differential system* if the maximum of the degrees of the polynomials P and Q is two.

Quadratic polynomial differential systems have been investigated intensively, and more than one thousand papers have been published about these systems (see for instance [**3, 15–17**]), but the problem of classifying all the integrable quadratic polynomial differential systems remains open. For more information on integrable differential systems in dimension two, see for instance [**5**].