J. Math. Anal. Appl. 379 (2011) 188–199

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Liouvillian first integrals of quadratic–linear polynomial differential systems

Jaume Llibre [a], Clàudia Valls [b,*]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
[b] Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001, Lisboa, Portugal

ARTICLE INFO

Article history:
Received 29 August 2010
Available online 21 December 2010
Submitted by Steven G. Krantz

Keywords:
Invariant algebraic curves
Darboux polynomials
Quadratic systems
Quadratic vector fields
Liouvillian integrability

ABSTRACT

For a large class of quadratic–linear polynomial differential systems with a unique singular point at the origin having non-zero eigenvalues, we classify the ones which have a Liouvillian first integral, and we provide the explicit expression of them.

1. Introduction

For planar differential systems the notion of integrability is based on the existence of a first integral. For such systems the existence of a first integral determines completely its phase portrait. Then a natural question arises: *Given a system of ordinary differential equations in* $\mathbb{R}^2$ *depending on parameters, how to recognize the values of such parameters for which the system has a first integral?*

In particular the planar integrable systems which are not Hamiltonian, i.e. the systems in $\mathbb{R}^2$ that cannot be written as $x' = -\partial H/\partial y$, $y' = \partial H/\partial x$ for some function $H : \mathbb{R}^2 \to \mathbb{R}$ of class C^2, are in general very difficult to detect. Here the prime denotes derivative with respect to the independent variable t.

The first step to detect those first integrals in different classes of functions, namely polynomial, rational, elementary or Liouvillian, is to determine the algebraic invariant curves (i.e., the so-called Darboux polynomials).

Let P and Q be two real polynomials in the variables x and y, then the system

$$x' = P(x, y), \qquad y' = Q(x, y), \tag{1}$$

is a *quadratic polynomial differential system* if the maximum of the degrees of the polynomials P and Q is two.

Quadratic polynomial differential systems have been investigated for many authors, and more than one thousand papers have been published about these systems (see for instance [14] and [16]), but the problem of classifying all the integrable quadratic polynomial differential systems remains open.

Let $U \subset \mathbb{R}^2$ be an open set. We say that the non-constant function $H : U \to \mathbb{R}$ is a first integral of the polynomial vector field X on U, if $H(x(t), y(t)) =$ constant for all values of t for which the solution $(x(t), y(t))$ of X is defined on U. Clearly H is a first integral of X on U if and only if $XH = 0$ on U.

* Corresponding author.
E-mail addresses: jllibre@mat.uab.cat (J. Llibre), cvalls@math.ist.utl.pt (C. Valls).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.12.033