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Polynomial, rational and analytic first integrals for a family of 3-dimensional Lotka-Volterra systems

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Abstract. We extend the study of the integrability done by Leach and Miritzis (J Nonlinear Math Phys 13:535–548, 2006) on the classical model of competition between three species studied by May and Leonard (SIAM J Appl Math 29:243–256, 1975), to all real values of the parameters. Additionally, our results provide all polynomial, rational and analytic first integrals of this extended model. We also classify all the invariant algebraic surfaces of these models.

Mathematics Subject Classification (2000). 34C05 · 34A34 · 34C14.

Keywords. Polynomial integrability · Rational integrability · Analytic integrability · Lotka-Volterra systems.

1. Introduction and statement of the main results

Nonlinear ordinary differential equations appear in many branches of applied mathematics and physics. For a 3-dimensional system, the existence of two first integrals whose gradients are linearly independent in $\mathbb{R}^3$ except perhaps in a zero Lebesgue measure set determines completely its phase portrait because the intersections of the invariant levels of these two first integrals determine the trajectories of the system. The knowledge of a unique first integral reduces the study of the dynamics of the system from dimension 3 to dimension 2. So, the study of the existence of first integrals is an important subject in the qualitative theory of differential equations. Many different methods have been used for studying the existence of first integrals of non-linear differential systems based on Noether symmetries [6], the Darboux theory of integrability [8,17], the Lie symmetries [1,23], the Painlevé analysis [3], the use of Lax pairs [12], the direct method [9,10], the linear compatibility analysis method [24], the Carlemann embedding procedure [7,2], the quasimonomial formalism [4], etc.

In this paper, we use the Darboux theory of integrability to study the existence of first integrals for the model used by May and Leonard [18] for studying the competition among three species. This model is

$$\begin{aligned} \dot{X} &= X(1 - X - aY - bZ), \\ \dot{Y} &= Y(1 - bX - Y - aZ), \\ \dot{Z} &= Z(1 - aX - bY - Z), \end{aligned} \tag{1}$$

where $a, b \in \mathbb{R}$ and the dot denotes derivative with respect to the time t. Note that we are interested in the integrability of system (1) for all real values of its parameters, and not only for their positive values which are the ones with biological meaning.

Doing the change of variables

$$x = Xe^{-t}, \quad y = Ye^{-t}, \quad z = Ze^{-t}, \quad s = e^t,$$