FIRST INTEGRALS OF DARBOUX TYPE FOR A FAMILY OF 3-DIMENSIONAL LOTKA-VOLTERRA SYSTEMS

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ABSTRACT. We provide all the first integrals of Darboux type for the system studied by Leach and Miritzis (J. Nonlinear Math. Phys. **13** (2006), 535–548) on the classical model of competition between three species considered by May and Leonard (SIAM J. Appl. Math. **29** (1975), 243–256).

1. Introduction and statement of the main results

In this paper we use the Darboux theory of integrability to study the existence of first integrals of Darboux type for the model used by May and Leonard [10] for studying the competition among three species. This model is

$$\begin{aligned}\dot{X} &= X(1 - X - aY - bZ),\\ \dot{Y} &= Y(1 - bX - y - aZ),\\ \dot{Z} &= Z(1 - aX - bY - Z),\end{aligned} \tag{1}$$

where $a, b \in \mathbb{R}$ and the dot denotes derivative with respect to the time t.

Doing the change of variables

$$x = Xe^{-t}, \quad y = Ye^{-t}, \quad z = Ze^{-t}, \quad s = e^t,$$

system (1) becomes

$$\begin{aligned}x' &= -x(x + ay + bz),\\ y' &= -y(bx + y + az),\\ z' &= -z(ax + by + z),\end{aligned} \tag{2}$$

where $a, b \in \mathbb{R}$ and here the prime denotes derivative with respect to the new time s.

Leach and Miritzis in [5] proved that system (2) has a first integral when either $a + b = 2$, or $a = b$, but in [5] it is unknown if for other values of the parameters a and b, system (2) has or not other first integrals. In [8] the authors showed for the case $a+b = -1$, system (2) has also a first integral. We also note that the existence of first integrals for system (2) imply the existence of invariants for system (1). Here an *invariant* is a first integral depending on the time.

The known first integrals are

$$H_1(x, y, z) = \frac{xyz}{(x + y + z)^3},$$

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if $a+b=2$,

$$H_2(x,y,z) = (xyz)^{2+a}\big((x-y)(x-z)(y-z)\big)^{-(2a+1)},$$

if $a=b\neq 1$,

$$H_3(x,y,z)=\frac{x}{y} \quad \text{and} \quad H_4(x,y,z)=\frac{x}{z},$$

if $a=b=1$,

$$H_2(x,y,z) \quad \text{and} \quad H_5(x,y,z) = x^2y^2 - x^2yz - xy^2z + x^2z^2 - xyz^2 + y^2z^2,$$

if $a=b=-1$, and finally,

$$H_6(x,y,z)=xyz$$

if $a+b=-1$.

Let $U\subset\mathbb{R}^3$ be an open subset. We say that the non–constant function $H\colon U\to\mathbb{R}$ is a *first integral* of the polynomial vector field

$$\mathcal{X} = \big(-x(x+ay+bz), -y(bx+y+az), -z(ax+by+z)\big), \tag{3}$$

on U associated to system (2), if $H(x(t),y(t),z(t))$ = constant for all values of t for which the solution $(x(t),y(t),z(t))$ of $\mathcal{X}$ is defined on U. Clearly H is a first integral of $\mathcal{X}$ on U if and only if $\mathcal{X}H=0$ on U.

When H is a polynomial we say that H is a *polynomial first integral*.

Our main results on the polynomial integrability of system (2) were obtained in [8] and are:

Theorem 1. *The unique polynomial first integrals of system* (2) *are*

(a) $H_2(x,y,z)$ *when* $b=a\neq 1$ *and either* $-(2+a)/(2a+1)$, *or* $-(2a+1)/(2+a)$ *is a nonnegative integer; and all the polynomial functions in the variable* H_2.

(b) $H_2(x,y,z)$ *and* $H_5(x,y,z)$ *when* $b=a=-1$*; and all the polynomial functions in the variables* H_2 *and* H_5.

(c) $H_6(x,y,z)=xyz$ *if* $a+b=-1$*; and all the polynomial functions in the variable* H_6. *Note that* $H_6(x,y,z)=H_2(x,y,z)$ *when* $a=b=-1/2$.

For proving our main result concerning the existence of first integrals of Darboux type we shall use the invariant algebraic surfaces of system (2). This is the basis of the Darboux theory of integrability. The Darboux theory of integrability works for real or complex polynomial ordinary differential equations. As it is explained for instance in [6] the study of complex invariant algebraic surfaces is necessary for obtaining all the real first integrals of a real polynomial differential system.

Let $\mathbb{C}[x,y,z]$ be the ring of all polynomials with coefficients in $\mathbb{C}$ and variables x,y and z. We say that $h=h(x,y,z)=0$ with $h(x,y,z)\in\mathbb{C}[x,y,z]\setminus\mathbb{C}$ is an *invariant algebraic surface* of the vector field $\mathcal{X}$ if it satisfies $\mathcal{X}h=Kh$ for some polynomial $K=K(x,y,z)\in\mathbb{C}[x,y,z]$, called the *cofactor* of $h=0$. Note that K has degree at most 1. The polynomial h is called a *Darboux polynomial*, and we also say that K is the *cofactor* of the Darboux polynomial h. We note that a Darboux polynomial with zero cofactor is a polynomial first integral.

In the next result obtained in [8] the authors characterize all the irreducible Darboux polynomials of system (2) with non-zero cofactor. Taking into account that system (2) is homogeneous, the study of the Darboux polynomials of system

(2) with nonzero cofactor can be reduced to the study of the homogeneous Darboux polynomials with nonzero cofactor, for more details see [14].

Theorem 2. *The unique Darboux polynomials of system* (2) *of degree* $n \geq 1$ *with nonzero cofactor are*

(a) $x^{n_1} y^{n_2} z^{n-n_1-n_2}$, *with* n_1, n_2 *integers satisfying* $0 \leq n_1 + n_2 \leq n$, *for all* $a, b \in \mathbb{R}$.

(b) $x^{n_1} y^{n_2} z^{n_3} (x+y+z)^{n-n_1-n_2-n_3}$ *with* n_1, n_2, n_3 *integers satisfying* $0 \leq n_1 + n_2 + n_3 \leq n$, *if* $a + b = 2$.

(c) $x^{n_1} y^{n_2} z^{n_3} (x-y)^{n_4} (y-z)^{n_5} (z-x)^{n-n_1-n_2-n_3-n_4-n_5}$ *with* n_1, n_2, n_3, n_4, n_5 *integers satisfying* $0 \leq n_1 + n_2 + n_3 + n_4 + n_5 \leq n$, *if* $b = a \neq 1$.

(d) $P(x, y)^n$ *with* P *any polynomial of degree* 1 *if* $a = b = 1$.

The integrability of other kind of 3-dimensional Lotka-Volterra systems different to system (2) has already been studied. See for instance [1, 4, 7, 11, 12, 13].

We say that $E = \exp(h/g)$, with $g, h \in \mathbb{C}[x, y, z]$ coprime and $E \notin \mathbb{C}$, is an *exponential factor* of the vector field $\mathcal{X}$ given in (3) if it satisfies $\mathcal{X}E = LE$ for some polynomial $L = L(x, y, z) \in \mathbb{C}[x, y, z]$, called the *cofactor* of E and having degree at most 1. The equality $\mathcal{X}E = LE$ is equivalent to

$$(4) \quad -x(x+ay+bz)\frac{\partial(h/g)}{\partial x} - y(bx+y+az)\frac{\partial(h/g)}{\partial y} - z(ax+by+z)\frac{\partial(h/g)}{\partial z} = L.$$

The exponential factors appear when an invariant algebraic surface has multiplicity larger than 1, for more details see [2] and [9].

A first integral H of system (2) is called of *Darboux type* if it is of the form

$$H = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q},$$

where $f_1, \ldots, f_p$ are Darboux polynomials, $F_1, \ldots, F_q$ are exponential factors and $\lambda_j, \mu_k \in \mathbb{C}$ for $j = 1, \ldots, p$, $k = 1, \ldots, q$.

It is well known that system (2) when $a = b = 1$ or $a = b = -1$ is completely integrable (i.e. it has two independent first integrals) with the first integrals H_3, H_4 when $a = b = 1$, and H_2, H_5 when $a = b = -1$. In what follows we do note consider these two cases.

The following is the main result of this paper.

Theorem 3. *The unique first integrals of Darboux type of system* (2) *except when* $a = b = \pm 1$ *are of the form* $H = G^\lambda \exp(G)^\mu$ *where* $\lambda, \mu \in \mathbb{C}$ *and*

(a) $G = (x+y+z)^3/(xyz)$ *if* $a + b = 2$;

(b) $G = ((x-y)(y-z)(z-x))^{2a+1}/(xyz)^{2+a}$ *if* $b = a$;

(c) $G = xyz$ *if* $a + b = -1$.

In Section 2 we state and prove preliminary results for the homogeneous polynomial differential system (2). In Section 3 we prove Theorem 3 for the case $a+b = 2$ and $(a, b) \neq \pm 1$. In section 4 we prove Theorem 3 for the case $b = a \neq \pm 1$ and finally, in Section 5 we prove Theorem 3 for the case $b \neq a$.

2. Preliminaries for system (2)

We start with a result about Darboux theory of integrability that we shall use in the paper.

Theorem 4. *Suppose that system* (2) *admits* p *invariant algebraic curves* $f_i = 0$ *with cofactors* K_i *for* $i = 1, \dots, p$ *and* q *exponential factors* $E_j = \exp(h_j/g_j)$ *with cofactors* L_j *for* $j = 1, \dots, q$. *Then there exist* $\lambda_j, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$$

if and only if the function $f_1^{\lambda_1} \cdots f_p^{\lambda_p} E_1^{\mu_1} \cdots E_q^{\mu_q}$ *is a first integral of system* (2).

Let $\mathbb{N}$ be the set of positive integers. For $n \in \mathbb{N}$ we define

$$C_n = \{a, b \in \mathbb{R} : a = 1 - n_1,\ b = 1 - n_2,\ n_1 n_2 = n,\ n_1 + n_2 \le n,\ n_1, n_2 \in \mathbb{N}\}.$$

In [8] the authors proved the following result.

Proposition 5. *For system* (2) *restricted to* $z = 0$ *the following statements hold.*

(a) *The unique polynomial first integrals are*

$$H = x^{1-b} y^{1-a} ((b-1)x + (1-a)y)^{-1+ab} \tag{5}$$

if $(a, b) \in C_n$; *and all polynomials in the variable* H.

(b) *All the irreducible Darboux polynomials with non–zero cofactor are* x *and* y *for all* a *and* b; *and additionally* $(b-1)x + (1-a)y$ *when* $a \neq 1$ *and* $b \neq 1$.

Proposition 6. *The following statements hold.*

(a) *If* $E = \exp(g/h)$ *is an exponential factor for the polynomial system* (2) *and* h *is not a constant polynomial, then* $h = 0$ *is an invariant algebraic curve.*

(b) *Eventually* $\exp(g)$ *can be an exponential factor, coming from the multiplicity of the infinite invariant straight line.*

For a geometrical meaning of the exponential factors and a proof of Proposition 6 see [2].

Let $\sigma \colon \mathbb{C}[x, y, z] \to \mathbb{C}[x, y, z]$ be the automorphism

$$\sigma(x) = y, \quad \sigma(y) = z, \quad \sigma(z) = x.$$

Proposition 7. *If* G *is an exponential factor for system* (2) *with cofactor* $L = a_0 + a_1 x + a_2 y + a_3 z$, *then* $F = G \cdot \sigma(G) \cdot \sigma^2(G)$ *is an exponential factor of system* (2) *invariant by* σ *with cofactor* $L = a_0 + \alpha(x+y+z)$, $a_0 \in \mathbb{C}$ *and* $\alpha = a_1 + a_2 + a_3 \in \mathbb{C} \setminus \{0\}$.

Proof. Since (2) is invariant under σ and σ^2, so $F = G \cdot \sigma(G) \cdot \sigma^2(G)$ is also a exponential factor of system (2) with cofactor $L + \sigma(L) + \sigma^2(L) = 3a_0 + \alpha(x+y+z)$, $a_0, \alpha = a_1 + a_2 + a_3 \in \mathbb{C}$. □

We denote by $\mathbb{N}$ the set of all positive integers.

Remark 8. *If the exponential factor* $e^{h/g}$ *invariant by* σ *has cofactor* L, *then* $L = \sigma(L) = \sigma^2(L)$, *and consequently* $L = a_0 + \alpha(x + y + z)$ *using the notation of Proposition 7.*

Proposition 9. *System* (2) *with* $a = b \neq \pm 1$ *restricted to* $z = x$ *has the Darboux polynomial* f *of degree* n *of the form* $x^{m_1} y^{m_2} (x - y)^{n - m_1 - m_2}$ *with* $m_1, m_2 \in \mathbb{N}$ *and* $m_1 + m_2 \le n$.

Proof. We introduce the change of variables $(X,Y,Z)=(x,y,z-x)$. In these variables system (2) becomes

$$X'=-X((1+a)X+aY+aZ),\quad Y'=-Y(2aX+Y+aZ),\quad Z'=-Z(2X+aY+Z). \tag{6}$$

We consider system (6) restricted to $Z=0$. It follows by direct computations that

$$G=\frac{Y^{-(1+a)}(Y-X)^{1+2a}}{X}$$

is a first integral of system (6) restricted to $Z=0$. In order that G is a polynomial we must have $1+a=m$ for some non-negative integer m, i.e. $a=m-1$ and $-1-2a$ also a non-negative integer. But $-1-2a=-1-2(m-1)=1-2m$ has to be a non-negative integer and this is only possible if $m=0$. In this case $a=-1$ and we do not consider this case here. Hence G is never a polynomial.

It also follows by direct computations that X,Y and $X-Y$ are the unique homogeneous Darboux polynomials of degree one of system (6) restricted to $Z=0$.

Let now f be a homogeneous irreducible Darboux polynomial of degree $n\geq 2$ of system (6) restricted to $Z=0$. The cofactor of f is of the form $K=b_0X+b_1Y$, with $b_0,b_1\in\mathbb{C}$. We assume that either $b_0\neq 0$ or $b_1\neq 0$. Then f satisfies

$$-X((1+a)X+aY)\frac{\partial f}{\partial X}-Y(2aX+Y)\frac{\partial f}{\partial Y}=(b_0X+b_1Y)f. \tag{7}$$

If we restrict equation (7) to $X=0$ and denote by $\overline{f}$ the restriction of f to $X=0$ we get that $\overline{f}\neq 0$ (otherwise f would be reducible) and $\overline{f}=\overline{f}(Y)$ is a homogeneous polynomial of degree n, that is $\overline{f}=\alpha_0Y^n$ with $\alpha_0\in\mathbb{C}\setminus\{0\}$. On the other hand $\overline{f}$ is a homogeneous Darboux polynomial of degree n of system (6) restricted to $X=Z=0$, that it satisfies

$$-Y^2\frac{d\overline{f}}{dY}=b_1Y\overline{f},\quad \text{i.e.}\quad \overline{f}=\alpha_0Y^{-b_1},\quad \alpha_0\in\mathbb{C}.$$

Therefore equating the two expressions for $\overline{f}$ we get $b_1=-n$. In a similar way, restricting to $Y=0$ we get $b_0=-n$. Thus $K=b_0X+b_1yY=-n(X+Y)$ and (7) becomes

$$-X((1+a)X+aY)\frac{\partial f}{\partial x}-Y(2aX+Y)\frac{\partial f}{\partial Y}=-n(X+Y)f. \tag{8}$$

Now we introduce the variables $(x_1,y_1)=(X,X-Y)$. Then system (6) restricted to $Z=0$ becomes

$$x_1'=-x_1((2a+1)x_1+ay_1),\quad y_1'=-y_1((a+2)x_1-y_1).$$

Let $\hat{f}=\hat{f}(x_1,y_1)=\overline{f}(X,Y)$. Now we denote by $\tilde{f}=\tilde{f}(y_1)=\hat{f}(0,y_1)$. Then $\tilde{f}$ satisfies (8) restricted to $x_1=0$, i.e,

$$y_1^2\frac{d\tilde{f}}{dy_1}=-y_1\tilde{f},\quad \text{that is}\quad \tilde{f}=\frac{\alpha}{y},\quad \alpha\in\mathbb{C}.$$

Since $\tilde{f}$ is a homogeneous polynomial of degree $n\geq 2$ we get a contradiction. This concludes the proof of the proposition. □

Lemma 10. *For an exponential factor $E=\exp(h/g)$ invariant by σ, the polynomial h satisfies $h=\sigma(h)$ and the polynomial g has the form*

(a) $(xyz)^m(x+y+z)^l$ *with $m,l\in\mathbb{N}$, if $a+b=2$;*

(b) $(xyz)^m((x-y)(y-z)(z-x))^l$ *with* $m, l \in \mathbb{N}$ *if* $a = b \neq 1$;
(c) $(xyz)^m$ *with* $m \in \mathbb{N}$ *for any other value of* a *and* b.

Proof. The exponential factor $E = \exp(h/g)$ satisfies $\sigma(E) = E$, i.e.

$$\exp(\sigma(h/g)) = \exp(\sigma(h)/\sigma(g)) = \exp(h/g).$$

Since h and g are coprime this implies that $\sigma(h) = h$ and $\sigma(g) = g$. We consider different cases.

Case 1: $a + b = 2$. In this case it follows from Theorems 1 and 2 that

$$\begin{aligned} g(x,y,z) &= x^{n_1} y^{n_2} z^{n_3} (x+y+z)^{n_4} = y^{n_1} z^{n_2} x^{n_3} (x+y+z)^{n_4}, \\ &= z^{n_1} x^{n_2} y^{n_3} (x+y+z)^{n_4}. \end{aligned}$$

Therefore $n_1 = n_2 = n_3 = m$ and $n_4 = l$. This completes the proof of the lemma in this case.

Case 2: $a = b \neq 1$. It follows from Theorems 1 and 2 that

$$\begin{aligned} g(x,y,z) &= x^{n_1} y^{n_2} z^{n_3} (x-y)^{n_4} (y-z)^{n_5} (z-x)^{n_6} \\ &= y^{n_1} z^{n_2} x^{n_3} (y-z)^{n_4} (z-x)^{n_5} (x-y)^{n_6} \\ &= z^{n_1} x^{n_2} y^{n_3} (z-x))^{n_4} (x-y)^{n_5} (y-z)^{n_6}. \end{aligned}$$

Thus $n_1 = n_2 = n_3 = m$ and $n_4 = n_5 = n_6 = l$. This completes the proof of the lemma in this case.

Case 3. $a \neq b$ and $a+b \neq 2$. Again from Theorems 1 and 2 $g(x,y,z) = x^{n_1} y^{n_2} z^{n_3}$. Proceeding as in the above two cases we obtain that $n_1 = n_2 = n_3 = m$. $\square$

Lemma 11. *The exponential factors of the form* e^h *invariant by* σ *have cofactor zero.*

Proof. Let L be the cofactor of e^h. From Proposition 7 the exponential factor $e^{h+\sigma(h)+\sigma^2(h)}$ has cofactor $3L = 3a_0 + \alpha(x+y+z)$. Since the exponential factor satisfies equation (4), taking in it $x = y = z = 0$ we get that $a_0 = 0$.

Now taking $x = y = 0$ in (4) and denoting by $\overline{h}$ the restriction of h to $x = y = 0$ we have that

$$-z^2 \frac{d\overline{h}}{dz} = \alpha z, \quad \text{that is} \quad h = C - \alpha \log|z|, \quad C \in \mathbb{C}.$$

Since h must be a polynomial we have that $\alpha = 0$. Therefore, $L = 0$. $\square$

3. Proof of Theorem 3 when $a + b = 2$ except $a = b = 1$

We divide the proof of Theorem 3 with $a + b = 2$ except $a = b = 1$ into different partial results.

Proposition 12. *System* (2) *with* $a + b = 2$ *except* $a = b = 1$ *has no exponential factor* E *invariant by* σ *with cofactor* $L = a_0 + \alpha(x+y+z)$ *with* $\alpha \neq 0$.

Proof. Let E be an exponential factor invariant by σ. By Lemma 10(a) it has the form $E = \exp(h/((xyz)^m(x+y+z)^l))$ with $\sigma(h) = h$ and $h \in \mathbb{C}[x,y,z]$. It follows from Lemma 11 that $m \neq 0$ or $l \neq 0$, otherwise $\alpha = 0$. Then h satisfies

(9)
$$\begin{aligned} &- x(x + ay + (2-a)z)\frac{\partial h}{\partial x} - y((2-a)x + y + az)\frac{\partial h}{\partial y} - z(ax + (2-a)y + z)\frac{\partial h}{\partial z} \\ &+ (3m + l)(x+y+z)h = (a_0 + \alpha(x+y+z))(xyz)^m(x+y+z)^l, \end{aligned}$$

where we have simplified the common factor $\exp(h/((xyz)^m(x+y+z)^l))$ and multiplied by $(xyz)^m(x+y+z)^l$. We consider different cases.

Case 1: $m \neq 0$. In this case h is coprime with xyz. Taking $z=0$ in (9) and denoting by $\overline{h}$ the restriction of h to $z=0$ we have that $\overline{h} \neq 0$ and satisfies

$$-x(x+ay)\frac{\partial \overline{h}}{\partial x} - y((2-a)x+y)\frac{\partial \overline{h}}{\partial y} + (3m+l)(x+y)\overline{h} = 0. \tag{10}$$

That is $\overline{h}$ is a Darboux polynomial of system (2) restricted to $z=0$. By Proposition 5(b) we have that $\overline{h}$ must be of the form $\overline{h} = \alpha_0 x^{n_1} y^{n_2} (x+y)^{n_3}$ with $\alpha_0 \in \mathbb{C}\setminus\{0\}$. Then imposing that it satisfies (10) we get

$$-n_1(x+ay) - n_2((2-a)x+y) - n_3(x+y) = -(3m+l)(x+y), \tag{11}$$

that is

$$-n_1 - (2-a)n_2 - n_3 = -(3m+l), \quad -an_1 - n_2 - n_3 = -(3m+l),$$

i.e.,

$$-n_1 - (2-a)n_2 = -an_1 - n_2,$$

or equivalently,

$$(a-1)n_1 = (1-a)n_2.$$

Since $a \neq 1$ and $n_1, n_2 \in \mathbb{N}\cup\{0\}$ we get that $n_1 = n_2 = 0$. From (11) $n_3 = 3m+l$. Hence $\overline{h} = \alpha_0(x+y)^{3m+l}$ with $\alpha_0 \in \mathbb{C}\setminus\{0\}$. Using the Newton's binomial formula, we can write h as

$$h = \alpha_0(x+y)^{3m+l} + zg_1 = \alpha_0(x+y+z)^{3m+l} + zg_2$$

for some polynomial $g_k = g_k(x,y,z)$ for $k=1,2$. Since h is invariant by σ, using that $h = \sigma(h) = \sigma^2(h)$, we have that g_2 can be written in the three equivalent forms

$$zg_2 = x\sigma(g_2) = y\sigma^2(g_2), \quad \text{and hence} \quad h = \alpha_0(x+y+z)^{3m+l} + xyzf$$

for some polynomial $f = f(x,y,z)$. Then

$$E = \exp\left(\frac{h}{(xyz)^m(x+y+z)^l}\right) = \exp\left(\alpha_0\frac{(x+y+z)^{3m}}{(xyz)^m}\right)G,$$

with

$$G = \exp\left(\frac{f}{(xyz)^{m-1}(x+y+z)^l}\right).$$

Since $(x+y+z)^{3m}/(xyz)^m$ is a first integral of system (2) we get that G and E satisfy the same equation (9) with h replaced by f and m replaced by $m-1$. Then proceeding $m-1$ times for G as we did for E we get that

$$E = \exp\left(\sum_{j=0}^{m}\alpha_j\frac{(x+y+z)^{3(m-j)}}{(xyz)^{m-j}} + \frac{(xyz)T}{(x+y+z)^l}\right),$$

with $\alpha_j \in \mathbb{C}$. Then from (9) since $(x+y+z)^3/(xyz)$ is a first integral of system (2) when $b = 2-a$, we get

(12)

$$xyz\Big(-x(x+ay+(2-a)z)\frac{\partial T}{\partial x} - y((2-a)x+y+az)\frac{\partial T}{\partial y} - z(ax+(2-a)y+z)\frac{\partial T}{\partial z} + (l-3)(x+y+z)T\Big) = (a_0 + \alpha(x+y+z))(x+y+z)^l.$$

This equality implies that $\alpha = 0$, a contradiction.

Case 2: $m = 0$. Then $l \neq 0$ and h satisfies

$$
\begin{aligned}
&- x(x + ay + (2-a)z)\frac{\partial h}{\partial x} - y((2-a)x + y + az)\frac{\partial h}{\partial y} - z(ax + (2-a)y + z)\frac{\partial h}{\partial z} \\
&+ l(x+y+z)h = (a_0 + \alpha(x+y+z))(x+y+z)^l.
\end{aligned}
$$

Evaluating this equation on $x = y = 0$ and denoting by $\overline{h}$ the restriction of h to $x = y = 0$, we have that $\overline{h}$ satisfies

$$
-z^2\frac{d\overline{h}}{dz} + lz\overline{h} = (a_0 + \alpha z)z^l.
$$

So we obtain that

$$
\overline{h} = z^l\Big(C + \frac{a_0}{z} - \alpha \log|z|\Big).
$$

Since $\overline{h}$ must be a polynomial we have that $\alpha = 0$, a contradiction. This concludes the proof of the proposition. □

Lemma 13. *If system* (2) *with* $a+b=2$ *except when* $a=b=1$ *has an exponential factor invariant by* σ *with cofactor* $L = 0$*, then* $\exp((x+y+z)^3/(xyz))$ *is an exponential factor.*

Proof. From (9) and since $a_0 = \alpha = 0$ the exponential factor $E = \exp(h/((xyz)^m(x+y+z)^l))$ satisfies

(13)

$$
\begin{aligned}
&- x(x + ay + (2-a)z)\frac{\partial h}{\partial x} - y((2-a)x + y + az)\frac{\partial h}{\partial y} - z(ax + (2-a)y + z)\frac{\partial h}{\partial z} \\
&+ (3m+l)(x+y+z)h = 0.
\end{aligned}
$$

Hence h is a Darboux polynomial invariant by σ of system (2) with $a+b=2$, except when $a=b=1$. In view of Theorem 2 it must be of the form $h = \alpha_0(xyz)^{n_1}(x+y+z)^{n_2}$, with $\alpha_0 \in \mathbb{C}\setminus\{0\}$. Substituting h in (13) we obtain that

$$
3n_1 + n_2 = 3m + l \quad \text{i.e.,} \quad n_2 = l + 3(m - n_1).
$$

Substituting h in E we get that $E = \big(\exp((x+y+z)^3/(xyz))\big)^{m-n_1}$ and the proof of the lemma is completed. □

Proof of Theorem 3(a). Assume that H is a first integral of Darboux type. First we consider that H is invariant by σ. Therefore the Darboux polynomials and the exponential factors which appear in the expression of H are also invariant by σ. Then, from Theorems 4, 1 and 2 and Proposition 12, H can be written as

$$
H = (xyz)^{\lambda_1}(x+y+z)^{\lambda_2}J^{\mu}, \quad \lambda_1, \lambda_2, \mu \in \mathbb{C},
$$

with $J = \exp(h/g)$ an exponential factor invariant by σ with cofactor $L = a_0$. Then we have

$$
-(3\lambda_1 + \lambda_2)(x+y+z) + \mu a_0 = 0.
$$

Therefore we get that $\lambda_2 = -3\lambda_1$ and $\mu a_0 = 0$. If $a_0 = 0$ then J is an exponential factor invariant by σ with $L = 0$. By Lemma 13 we get that $J = \exp((x+y+z)^3/(xyz))$ and $H = \big((x+y+z)^3/(xyz)\big)^{\lambda_1}J^{\mu}$. If $\mu = 0$ then $H = \big((x+y+z)^3/(xyz)\big)^{\lambda_1}$. This completes the proof of the theorem in this case.

Now we assume that the first integral of Darboux type H is not invariant by σ. Then $H\sigma(H)\sigma^2(H)$ is another first integral which is invariant by σ, and hence from the first part of this proof this first integral is of the form $(x+y+z)^3/(xyz)J^{\mu/\lambda}$, with $J = \exp((x+y+z)^3/(xyz))$. That is

$$H\sigma(H)\sigma^2(H) = (x+y+z)^3/(xyz)J^{\mu/\lambda}. \tag{14}$$

Clearly there exists $\alpha > 0$ such that either J^α divides H, $\sigma(H)$ or $\sigma^2(H)$. Since J is invariant by σ it must divide H, $\sigma(H)$ and $\sigma^2(H)$, then

$$H = J^{\mu/(3\lambda)}H_1, \quad \sigma(H) = J^{\mu/(3\lambda)}\sigma(H_1), \quad \sigma^2(H) = J^{\mu/(3\lambda)}\sigma^2(H_1),$$

where H_1 is not invariant by σ. In view of (14) and since $x+y+z$ is invariant by σ we have that either $(x+y+z)/x$ divides either H_1 or $\sigma(H_1)$ or $\sigma^2(H_1)$. Without loss of generality we can assume that divides H_1. Then

$$\begin{aligned} H &= \frac{x+y+z}{x}J^{\mu/(3\lambda)}H_2, \quad \sigma(H) = \frac{x+y+z}{y}J^{\mu/(3\lambda)}\sigma(H_2), \\ \sigma^2(H) &= \frac{x+y+z}{z}J^{\mu/(3\lambda)}\sigma^2(H_2). \end{aligned} \tag{15}$$

Therefore, from (14) and (15) we have that in fact $H = (x+y+z)J^{\mu/(3\lambda)}/x$, a contradiction with the fact that H is a first integrals. This completes the proof of the theorem. $\square$

4. Proof of Theorem 3 when $b = a \neq \pm 1$

We introduce some auxiliary results that we shall use in the proof.

Lemma 14. *Let $T = T(x,y,z)$ be a polynomial of degree $6l$ satisfying*

(16)
$$\begin{aligned} &-x(x+ay+az)\frac{\partial T}{\partial x} - y(ax+y+az)\frac{\partial T}{\partial y} - z(ax+ay+z)\frac{\partial T}{\partial z} \\ &+ 3(a+1)l(x+y+z)T = \frac{\alpha}{3}(x+y+z)(xyz(x-y)(x-z)(y-z))^l, \quad \alpha \in \mathbb{C}, \ l \geq 1. \end{aligned}$$

Then $\alpha = 0$.

Proof. Let $\overline{T} = \overline{T}(x,y) = T(x,y,0)$. Then $\overline{T}$ satisfies (16) restricted to $z=0$, that is

$$-x(x+ay)\frac{\partial \overline{T}}{\partial x} - y(ax+y)\frac{\partial \overline{T}}{\partial y} + 3(a+1)l(x+y)\overline{T} = 0.$$

Then $\overline{T}$ is either zero or a Darboux polynomial of system (2) restricted to $z=0$ and with cofactor $-3(a+1)l(x+y)$. We want to show that in this last case is not possible. We proceed by contradiction. In view of Proposition 5(b) we have

$$\overline{T} = \alpha_0 x^{n_1} y^{n_2} (x-y)^{6l-n_1-n_2}, \quad \alpha_0 \in \mathbb{C} \setminus \{0\},$$

and the cofactor is3

$$K = -3(a+1)l(x+y) = -6l(x+y) - (a-1)(n_2 x + n_1 y).$$

Using that $a \neq 1$ we get $n_1 = n_2 = 3l$. Therefore $\overline{T} = \alpha_0 x^{3l}y^{3l}$ with $\alpha_0 \in \mathbb{C}\setminus\{0\}$. Since T is invariant by σ, we get

$$T = \alpha_0 x^{3l}y^{3l} + zT_1 = \alpha_0 y^{3l}z^{3l} + x\sigma(T_1) = \alpha_0 z^{3l}x^{3l} + y\sigma^2(T_1),$$

that is

$$T = \alpha_0 x^{3l} y^{3l} + \alpha_0 y^{3l} z^{3l} + \alpha_0 z^{3l} x^{3l} + xyzS, \tag{17}$$

where $S = S(x, y, z)$ is a polynomial of degree $6l - 3$.

Now if we restrict (16) to $z = x$ ad we denote by $\hat{T}$ the restriction of T to $z = x$, we have

$$-x((1+a)x + ay)\frac{\partial \hat{T}}{\partial x} - y(2ax + y)\frac{\partial \hat{T}}{\partial y} + 3(a+1)l(2x+y)\hat{T} = 0,$$

i.e., $\hat{T}$ is either zero or is a Darboux polynomial of system (2) restricted to $z = x$ and with cofactor $K = -3l(a+1)(2x+y)$. In this second case we will arrive to a contradiction. Indeed, by Proposition 9 we get

$$\hat{T} = \alpha_1 x^{n_1} y^{n_2} (x-y)^{6l-n_1-n_2},$$

and the cofactor is

$$K = -6l(1+a)x - 6ly - (a-1)n_2 x - (a-1)n_1 y.$$

Then equating the two cofactors we obtain

$$(a-1)n_2 x = 0 \quad \text{and} \quad (a-1)n_1 y = 3l(a-1)y,$$

that is, since $a \neq 1$, we get $n_2 = 0$ and $n_1 = 3l$. Then $\hat{T} = \alpha_1 x^{3l}(x-y)^{3l}$, and thus $T = \alpha_1 x^{3l}(x-y)^{3l} + (z-x)T_1$, where $T_1 = T_1(x,y,z)$ is a polynomial of degree $6l-1$. Then since T is invariant by σ we get

$$\begin{aligned} T &= \alpha_1 x^{3l}(x-y)^{3l} + (z-x)T_1 = \alpha_1 y^{3l}(y-z)^{3l} + (x-y)\sigma(T_1) \\ &= \alpha_1 z^{3l}(z-x)^{3l} + (y-z)\sigma^2(T_1), \end{aligned}$$

that implies

$$T = \alpha_1 x^{3l}(x-y)^{3l} + \alpha_1 y^{3l}(y-z)^{3l} + \alpha_1 z^{3l}(z-x)^{3l} + (x-y)(y-z)(z-x)T_3, \tag{18}$$

where $T_3 = T_3(x,y,z)$ is a polynomial of degree $6l-3$. Equating (17) and (18) with $z = 0$ and $y = x$ we get

$$\alpha_0 x^{6l} = \alpha_1 x^{6l}, \quad \text{that is} \quad \alpha_1 = \alpha_0,$$

Now evaluating them on $z = y = 0$ we get

$$0 = \alpha_1 x^{6l} \quad \text{that is} \quad \alpha_1 = 0 \quad \text{and hence} \quad \alpha_0 = 0,$$

a contradiction. Hence $\overline{T} = \hat{T} = 0$. Therefore T is divisible by z. Since T is invariant by σ and σ^2 we get that T is divisible by xyz. Moreover, T must be divisible by $x - z$ and again since it is invariant by σ and σ^2 we get that it must e divisible by $(x-y)(y-z)(z-x)$. Therefore

$$T = xyz(x-y)(y-z)(z-x)S,$$

and $S \neq 0$ is a polynomial of degree $6(l-1)$ satisfying

$$\begin{aligned} &- x(x+ay+az)\frac{\partial S}{\partial x} - y(ax+y+az)\frac{\partial S}{\partial y} - z(ax+ay+z)\frac{\partial S}{\partial z} \\ &+ 3(a+1)(l-1)(x+y+z)S = \frac{\alpha}{3}(x+y+z)\big(xyz(x-y)(y-z)(z-x)\big)^{l-1}. \end{aligned}$$

Then proceeding $l-1$ times for S as we did for T, we obtain that $T = \alpha_2\big(xyz(x-y)(x-z)(y-z)\big)^l$ with $\alpha_2 \in \mathbb{C}\setminus\{0\}$ and satisfying

$$\alpha(x+y+z) = 0.$$

Then $\alpha = 0$. This concludes the proof of the lemma. $\square$

Proposition 15. *System* (2) *with* $b = a \neq \pm 1$ *has no exponential factors invariant by* σ *with cofactor* $L = a_0 + \alpha(x+y+z)$ *with* $\alpha \neq 0$.

Proof. Let E be an exponential factor of system (2) with $b = a \neq \pm 1$ invariant by σ. In this case by Lemma 10(b) $E = \exp(h/((xyz)^m((x-y)(y-z)(z-x))^l)$ with $\sigma(h) = h$ and $h \in \mathbb{C}[x,y,z]$ that satisfies

$$\begin{aligned}
&-x(x+ay+az)\frac{\partial h}{\partial x} - y(ax+y+az)\frac{\partial h}{\partial y} - z(ax+ay+z)\frac{\partial h}{\partial z}\\
&+((2a+1)m+(2+a)l)(x+y+z)h\\
&= (a_0+\alpha(x+y+z))(xyz)^m((x-y)(y-z)(z-x))^l,
\end{aligned}\tag{19}$$

where we have simplified the common factor $\exp(h/((xyz)^m((x-y)(y-z)(z-x))^l))$ and multiplied by $(xyz)^m((x-y)(y-z)(z-x))^l$.

We write $h = \sum_{j\geq 1} h_j$, where each h_j is a homogeneous polynomial of degree j. Then computing the terms of degree $3(m+l)+1$ in (19) we have

$$\begin{aligned}
&-x(x+ay+az)\frac{\partial h_{3(m+l)}}{\partial x} - y(ax+y+az)\frac{\partial h_{3(m+l)}}{\partial y} - z(ax+ay+z)\frac{\partial h_{3(m+l)}}{\partial z}\\
&+((2a+1)m+(2+a)l)(x+y+z)h_{3(m+l)}\\
&= \frac{\alpha}{3}(x+y+z)(xyz)^m((x-y)(y-z)(z-x))^l.
\end{aligned}\tag{20}$$

We note that $m^2+l^2 \neq 0$ otherwise the result follows from Lemma 11 and that $h_{3(m+l)} \neq 0$ since otherwise the result holds immediately from (20). We consider different cases.

Case 1*:* $m > l$. In this case we have that $m \neq 0$. Now if we restrict (20) to $z = 0$ and denote by $\overline{h}_{3(m+l)} = \overline{h}_{3(m+1)}(x,y) = h_{3(m+l)}(x,y,0)$ we get

$$\begin{aligned}
&-x(x+ay)\frac{\partial \overline{h}_{3(m+l)}}{\partial x} - y(ax+y)\frac{\partial \overline{h}_{3(m+l)}}{\partial y} + ((2a+1)m+(2+a)l)(x+y)\overline{h}_{3(m+l)}\\
&= 0.
\end{aligned}$$

That is $\overline{h}_{3(m+l)}$ is either 0 or a Darboux polynomial of system (2) restricted to $z = 0$ and with cofactor $-((2a+1)m+(2+a)l)(x+y)$. We will show by contradiction that this last case is not possible. Indeed, in view of Proposition 5(b) we have

$$\overline{h}_{3(m+l)} = \alpha_0 x^{n_1} y^{n_2}(x-y)^{3(m+l)-n_1-n_2}, \quad \alpha_0 \in \mathbb{C}\setminus\{0\},$$

and the cofactor is

$$K = -3(l+m)(x+y) - (a-1)(n_2x+n_1y).$$

Equating the two cofactors and since $a \neq 1$ we get $n_2 = n_1 = l+2m$. Then $3(m+l)-n_1-n_2 = l-m < 0$ which implies that $\overline{h}_{3(m+l)}$ is not a polynomial, a contradiction. Therefore $\overline{h}_{3(m+l)} = 0$, and since $h_{3(m+l)}$ is invariant by σ we get that $h_{3(m+l)} = xyzg$, where $g = g(x, y, z)$ is a homogeneous polynomial of degree $3(m + l - 1)$ and invariant by σ that satisfies after simplifying by xyz,

$$\begin{aligned}
&- x(x + ay + bz)\frac{\partial g}{\partial x} - y(bx + y + az)\frac{\partial g}{\partial y} - z(ax + by + z)\frac{\partial g}{\partial z}\\
&+ ((2 + a)l + (2a + 1)(m - 1))(x + y + z)g\\
&= \frac{\alpha}{3}(x + y + z)(xyz)^{m-1}((x - y)(x - z)(y - z))^{l}.
\end{aligned}$$

If $m - 1 > l$, proceeding for g as we did for $h_{3(m+l)}$ we get that $g = (xyz)f$, where $f = f(x, y, z)$ is a homogeneous polynomial of degree $3(m + l - 2)$ invariant by σ. Proceeding inductively $m - l - 1$ times we have $h_{3(m+l)} = \alpha_0(xyz)^{m-l}T$, with $\alpha_0 \in \mathbb{C}$ and $T = T(x, y, z)$ a polynomial of degree $6l$ and satisfying (16). Then by Lemma 14 we get $\alpha = 0$.

Case 2*:* $m = l$. In this case, we are under the assumptions of Lemma 14 with $h = T$ and thus $\alpha = 0$.

Case 3*:* $m < l$. We have that $l \neq 0$. Now if we restrict (20) to $z = x$ and denote by $\hat{h}_{3(m+l)}$ the restriction of $h_{3(m+l)}$ to $z = x$ we obtain

$$\begin{aligned}
&- x((1 + a)x + ay)\frac{\partial \hat{h}_{3(m+l)}}{\partial x} - y(2ax + y)\frac{\partial \hat{h}_{3(m+l)}}{\partial y}\\
&+ ((2a + 1)m + (2 + a)l)(2x + y)\hat{h}_{3(m+l)} = 0.
\end{aligned}$$

Then either $\hat{h}_{3(m+l)} = 0$ or $\hat{h}_{3(m+l)}$ is a Darboux polynomial of system (2) restricted to $z = x$ and with cofactor $K = -((2a+1)m + (2+a)l)(x + y)$. We will show that this last case is not possible. Hence, in view of Proposition 9 we have

$$\hat{h}_{3(m+l)} = \alpha_1 x^{n_1} y^{n_2}(x - y)^{3(m+l)-n_1-n_2}, \quad \alpha_1 \in \mathbb{C} \setminus \{0\},$$

and the cofactor is

$$K = -(3l + 3al + 3m + 3am - n_2 + an_2)x - (3l + 3m - n_1 + an_1)y.$$

Equating the two cofactors we get $n_1 = l + 2m$ and $n_2 = m - l$. Since $m < l$ we have that $\hat{h}_{3(m+l)}$ is not a polynomial, a contradiction. Therefore, $\hat{h}_{3(m+l)} = 0$ and thus $h_{3(m+l)}$ must be divisible by $z - x$. Since $h_{3(m+l)}$ is invariant by σ we get that it is in fact of the form

$$h_{3(m+l)} = (x - y)(y - z)(z - x)g,$$

where $g = g(x, y, z)$ is a homogeneous polynomial of degree $3(m+l-1)$ and invariant by σ that satisfies

$$\begin{aligned}
&- x(x + ay + az)\frac{\partial g}{\partial x} - y(ax + y + az)\frac{\partial g}{\partial y} - z(ax + ay + z)\frac{\partial g}{\partial z}\\
&+ ((2 + a)(l - 1) + (2a + 1)m)(x + y + z)g\\
&= \frac{\alpha}{3}(x + y + z)(xyz)^{m}((x - y)(y - z)(z - x))^{l-1}.
\end{aligned}$$

If $l-1 > m$, proceeding for g as we did for $h_{3(m+l)}$ we obtain $g = (x-y)(y-z)(z-x)f$ where $f = f(x, y, z)$ is a homogeneous polynomial of degree $3(m + l - 2)$ invariant by σ. Proceeding inductively $l - m - 1$ times we have $h_{3(m+l)} = \alpha_1\big((x - y)(y - z)(z - x)\big)^{l-m}T$ with $\alpha_1 \in \mathbb{C}$ and $T = T(x, y, z)$ a polynomial of degree $6m$ and satisfying (16) replacing l by m. Then by Lemma 14 we get that $\alpha = 0$. This concludes the proof of the proposition. □

Lemma 16. *If system* (2) *with* $b = a \neq \pm 1$, *has an exponential factor* E *invariant by* σ *with cofactor* $L = 0$, *then* $E = \exp((x-y)(y-z)(z-x))^{2a+1}/(xyz)^{2+a})$.

Proof. By Lemma 10(b) the exponential factor $E = \exp(h/((xyz)^m((x-y)(y-z)(z-x))^l))$ satisfies

$$\begin{aligned}&-x(x+ay+bz)\frac{\partial h}{\partial x} - y(bx+y+az)\frac{\partial h}{\partial y} - z(ax+by+z)\frac{\partial h}{\partial z}\\&+((2+a)l+(2a+1)m)(x+y+z)h = 0.\end{aligned}$$

In view of Theorem 2(c) we obtain

$$h = (xyz)^{n_1}((x-y)(y-z)(z-x))^{n_2},$$

with cofactor

$$-((2+a)n_2 + (2a+1)n_1)(x+y+z) = -((2+a)l + (2a+1)m)(x+y+z).$$

That is $E = \exp(((x-y)(y-z)(z-x))^{n_2-l}/(xyz)^{m-n_1})$, with

$$(2+a)n_2 + (2a+1)n_1 = (2+a)l + (2a+1)m.$$

Then if $a \neq -2$, $n_2 - l = (2a+1)(m-n_1)/(2+a)$ and $E = \exp((x-y)(y-z)(z-x))^{2a+1}/(xyz)^{2+a})^{m-n_1}$. If $a = -2$ then $n_1 = m$ and consequently $E = \exp((x-y)(y-z)(z-x))^{n_2-l}$. In short the proposition is proved. □

Proof of Theorem 3(b). Proceeding in a similar way as in the proof of Theorem 3(a) we first assume that the first integral of Darboux type is invariant by σ. From Theorems 4, 1 and 2 and Proposition 12, if system (2) has a Darboux first integral H (that again can be considered invariant by σ) then

$$H = (xyz)^{\lambda_1}((x-y)(y-z)(z-x))^{\lambda_2} J^{\mu}, \quad \lambda_1, \lambda_2, \mu \in \mathbb{C},$$

with $J = \exp(h/g)$ an exponential factor invariant by σ with cofactor $L = a_0$. Then we have

$$-((2+a)\lambda_2 + (2a+1)\lambda_1)(x+y+z) + \mu a_0 = 0. \tag{21}$$

Therefore we get that $(2+a)\lambda_2 + (2a+1)\lambda_1 = 0$ and $\mu a_0 = 0$. If $a_0 = 0$ then J is an exponential factor invariant by σ with $L = 0$. By Lemma 16 we get that $J = \exp((x-y)(y-z)(z-x))^{2a+1}/(xyz)^{2+a})$ and

$$H = \left(\frac{((x-y)(y-z)(z-x))^{2a+1}}{(xyz)^{2+a}}\right)^{\lambda_1} J^{\mu}.$$

If $\mu = 0$ then

$$H = \left(\frac{((x-y)(y-z)(z-x))^{2a+1}}{(xyz)^{2+a}}\right)^{\lambda_1}.$$

Now the same arguments that we did in the proof of Theorem 3(a) would allow to show that the unique first integrals of Darboux type of system (2) are invariant by σ. This completes the proof of the theorem. □

5. Proof of Theorem 3 when $a+b\neq 2$ and $b\neq a$

We introduce some auxiliary results.

Proposition 17. *System (2) with $a+b\neq 2$ and $b\neq a$ has no exponential factors E invariant by σ with cofactor $L=a_0+\alpha(x+y+z)$ with $\alpha\neq 0$.*

Proof. By Lemma 10(c) $E=\exp(h/(xyz)^m)$ with $\sigma(h)=h$ and $h\in\mathbb{C}[x,y,z]$ and satisfies

$$
\begin{aligned}
&-x(x+ay+bz)\frac{\partial h}{\partial x}-y(bx+y+az)\frac{\partial h}{\partial y}-z(ax+by+z)\frac{\partial h}{\partial z}+\\
&m(1+a+b)(x+y+z)h=(a_0+\alpha(x+y+z))(xyz)^m,
\end{aligned}
\tag{22}
$$

where we have simplified the common factor $\exp(h/(xyz)^m)$ and multiplied by $(xyz)^m$. We have $m\neq 0$, otherwise the result follows directly from Lemma 11. We consider different cases.

Case 1: $a+b+1=0$. In this case we have

$$
\begin{aligned}
&-x(x+ay-(1+a)z)\frac{\partial h}{\partial x}-y(-(1+a)x+y+az)\frac{\partial h}{\partial y}-z(ax-(1+a)y+z)\frac{\partial h}{\partial z}\\
&=(a_0+\alpha(x+y+z))(xyz)^m.
\end{aligned}
\tag{23}
$$

Now taking $z=0$ in (23) and denoting by $\overline{h}=\overline{h}(x,y)=h(x,y,0)$ we have that

$$
-x(x+ay)\frac{\partial \overline{h}}{\partial x}+y(-(1+a)x+y)\frac{\partial \overline{h}}{\partial y}=0,
$$

that is, $\overline{h}=0$ or $\overline{h}$ is a polynomial first integral of system (2) restricted to $z=0$. Since h is coprime with xyz, we get that $\overline{h}\neq 0$ and thus it is a Darboux polynomial of system (2) restricted to $z=0$. By Proposition 5(a) we have that $a=1-n_1$, $b=1-n_2$ with $n_1,n_2\in\mathbb{N}$, but then $0=a+b+1=3-n_1-n_2$, i.e. $n_1+n_2=3$. By Proposition 5(a) the first integral is $H=x^{n_2}y^{n_1}((b-1)x+(1-a)y)^{n-n_1-n_2}$ and $n\geq 3$ with $n_1n_2=n$ and $n_1+n_2=3$, a contradiction. Hence this case is not possible.

Case 2: $a\neq 1$, $b\neq 1$ and $1+a+b\neq 0$. We write $h=\sum_{j\geq 1}h_j$, where each h_j is a homogeneous polynomial of degree j. Computing the terms of degree $3m+1$ in (22) we have

$$
\begin{aligned}
&-x(x+ay+bz)\frac{\partial h_{3m}}{\partial x}-y(bx+y+az)\frac{\partial h_{3m}}{\partial y}-z(ax+by+z)\frac{\partial h_{3m}}{\partial z}+\\
&m(1+a+b)(x+y+z)h_{3m}=\alpha(x+y+z)(xyz)^m.
\end{aligned}
\tag{24}
$$

Now if we restrict (24) to $z=0$ and denote by $\overline{h}_{3m}$ the restriction of h_{3m} to $z=0$ we obtain

$$
-x(x+ay)\frac{\partial \overline{h}_{3m}}{\partial x}-y(bx+y)\frac{\partial \overline{h}_{3m}}{\partial y}+m(1+a+b)(x+y)\overline{h}_{3m}=0.
$$

That is $\overline{h}_{3m}$ is either 0 or a Darboux polynomial of system (2) restricted to $z=0$ and with cofactor $-m(1+a+b)(x+y)$. We will show that this last case is not possible. In view of Proposition 5(b) we have

$$
\overline{h}_{3m}=\alpha_0x^{n_1}y^{n_2}((b-1)x+(1-a)y)^{3m-n_1-n_2},\quad \alpha_0\in\mathbb{C}\setminus\{0\},
$$

and the cofactor is

$$K = ((1-b)n_2 - 3m)x + ((1-a)n_1 - 3m)y.$$

Then equating the cofactors we get

$$n_1 = \frac{m(a+b-2)}{a-1} = m + \frac{m(b-1)}{a-1}, \quad n_2 = \frac{m(a+b-2)}{b-1} = m + \frac{m(a-1)}{b-1}.$$

Note that

$$\frac{n_1}{n_2} = \frac{b-1}{a-1} > 0.$$

In order that $\overline{h}_{3m}$ be a polynomial we must have $n_1 + n_2 \leq 3m$. Then

$$3m \geq n_1 + n_2 = 2m + m\Big(\frac{b-1}{a-1} + \frac{a-1}{b-1}\Big),$$

that is

$$m \geq m\Big(\frac{(b-1)^2 + (a-1)^2}{(b-1)(a-1)}\Big) \quad \text{i.e.} \quad (b-1)(a-1) \geq (b-1)^2 + (a-1)^2,$$

a contradiction since $a \neq 1$ and $b \neq 1$. Then $\overline{h}_{3m} = 0$ and thus h_{3m} must be divisible by z. Since h_{3m} is invariant by σ we get that $h_{3m} = xyzg$, where $g = g(x,y,z)$ is a homogeneous polynomial of degree $3(m-1)$ and invariant by σ. It satisfies after simplifying by xyz,

$$\begin{aligned} &- x(x+ay+bz)\frac{\partial g}{\partial x} - y(bx+y+az)\frac{\partial g}{\partial y} - z(ax+by+z)\frac{\partial g}{\partial z} \\ &+ (m-1)(1+a+b)(x+y+z)g = \frac{\alpha}{3}(x+y+z)(xyz)^{m-1}. \end{aligned}$$

Proceeding for g as we did for h_{3m} we get that $g = (xyz)f$, where $f = f(x,y,z)$ is a homogeneous polynomial of degree $3(m-2)$ invariant by σ. Proceeding inductively $m-2$ times we have $h_{3m} = \alpha_0(xyz)^m$, with $\alpha_0 \in \mathbb{C}$. Introducing it in (24) we obtain that

$$0 = \alpha(x+y+z), \quad \text{that is} \quad \alpha = 0.$$

Case 3*: $a = 1$, $b \neq 1$ and $2+b \neq 0$.* In this case we have

$$\begin{aligned} &- x(x+y+bz)\frac{\partial h}{\partial x} - y(bx+y+z)\frac{\partial h}{\partial y} - z(x+by+z)\frac{\partial h}{\partial z} + \\ &m(2+b)(x+y+z)h = (a_0 + \alpha(x+y+z))(xyz)^m. \end{aligned} \tag{25}$$

Now taking $z = 0$ in (25) and denoting by $\overline{h} = \overline{h}(x,y) = h(x,y,0)$ we have that

$$-x(x+y)\frac{\partial \overline{h}}{\partial x} - y(bx+y)\frac{\partial \overline{h}}{\partial y} + m(2+b)(x+y)\overline{h} = 0,$$

that is, $\overline{h} = 0$ or $\overline{h}$ is a Darboux polynomial of system (2) restricted to $z = 0$. Since h is coprime with xyz, we get that $\overline{h} \neq 0$ and thus it is a Darboux polynomial of system (2) restricted to $z = 0$ with $a = 1$, $b + 2 \neq 0$ and with cofactor $K = -m(2+b)(x+y)$. In view of Proposition 5(b) it must be of the form $\overline{h} = \alpha_0 x^{n_1} y^{n_2}$, $\alpha_0 \in \mathbb{C} \setminus \{0\}$. The cofactor is

$$K = -(n_1 + n_2 b)x - (n_1 + n_2)y.$$

Equating the two cofactors K for $\overline{h}$ we get

$$n_1 + n_2 = m(2+b) \quad n_1 + n_2 b = m(2+b).$$

That is

$$n_2(b-1)=0 \quad \text{i.e.,} \quad n_2=0 \quad \text{and} \quad n_1=m(2+b).$$

Therefore $h = \alpha_0 x^{m(2+b)} + zg$ for some polynomial $g = g(x,y,z)$. Since h is invariant by σ we have

$$h=\alpha_0 x^{m(2+b)}+zg=\alpha_0 y^{m(2+b)}+x\sigma(g)=\alpha_0 z^{m(2+b)}+y\sigma^2(g), \quad \alpha_0\in\mathbb{C}.$$

In particular, on $z=0$ we obtain

$$\alpha_0 x^{m(2+b)}=y\sigma^2(g),$$

which implies $\alpha_0=0$, a contradiction. Therefore this case is not possible.

Case 4: $b=1$, $a\neq 1$ *and* $2+a\neq 0$. Proceeding as in Case 2 we have that this case is not possible. ☐

Proof of Theorem 3(c). We first assume that the first integral of Darboux type is invariant by σ. We can consider that $a+b\neq 2$ and $b\neq a$. We separate the proof into different cases.

Case 1: $a+b+1=0$. We first claim that if E is an exponential factor invariant by σ with cofactor $L=0$, then $E=\exp(xyz)^{\mu}$ for some constant μ. By Lemma 10(c) we have $E=\exp(h/((xyz)^m)$ and from (22) it satisfies

$$-x(x+ay+bz)\frac{\partial h}{\partial x}-y(bx+y+az)\frac{\partial h}{\partial y}-z(ax+by+z)\frac{\partial h}{\partial z}=0.$$

Then in view of Theorem 1(c) we have that h is a polynomial in the variable xyz. Therefore $E=\exp((xyz))^{\mu}$, and the claim is proved.

From Theorems 4, 1 and 2 and Proposition 17, if system (2) has a Darboux first integral invariant bby σ H then

$$H=(xyz)^{\lambda_1}J^{\mu}, \quad \lambda,\mu\in\mathbb{C}$$

with $J=\exp(h/g)$ an exponential factor invariant by σ with cofactor $L=a_0$. Then we have $\mu a_0=0$. If $a_0=0$ then J is an exponential factor invariant by σ with $L=0$. By the above explanation we have $J=\exp(xyz)$ and $H=(xyz)^{\lambda_1}J^{\mu}$. If $\mu=0$ then $H=(xyz)^{\lambda_1}$.

Case 2: $a+b+1\neq 0$. In this case we first show that there are no exponential factors invariant by σ with cofactor $L=0$. Indeed, by Lemma 10(c) we have $E=\exp(h/((xyz)^m)$ and satisfies

$$\begin{aligned}&-x(x+ay+bz)\frac{\partial h}{\partial x}-y(bx+y+az)\frac{\partial h}{\partial y}-z(ax+by+z)\frac{\partial h}{\partial z}+\\&m(1+a+b)(x+y+z)h=0.\end{aligned}$$

In view of Theorem 2(a) we obtain $h=(xyz)^{n_1}$, with cofactor

$$-n_1(1+a+b)(x+y+z)=-m(1+a+b)(x+y+z).$$

That is $E=\exp(\text{constant})$, which is not possible.

From Theorems 4, 1 and 2 and Proposition 17, if system (2) has a Darboux first integral H (that again can be considered invariant by σ) then

$$H=(xyz)^{\lambda_1}J^{\mu}, \quad \lambda,\mu\in\mathbb{C}$$

with $J = \exp(h/g)$ an exponential factor invariant by σ with cofactor $L = a_0$. We have

$$(1 + a + b)\lambda_1(x + y + z) + \mu a_0 = 0. \tag{26}$$

Therefore we get $\lambda_1 = 0$ and $\mu a_0 = 0$. If $a_0 = 0$ then J is an exponential factor invariant by σ with $L = 0$ and by the explanation above this is not possible. Then $\mu = 0$ and H is constant, a contradiction. Hence this case is not possible.

Now the same arguments that we did in the proof of Theorem 3(a) would allow to show that the unique first integrals of Darboux type of system (2) are invariant by σ. This completes the proof of the theorem. □

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