A note on the first integrals of the ABC system

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Without loss of generality the ABC systems reduce to two cases: either $A = 0$ and B, $C \geq 0$, or $A = 1$ and $0 < B, C \leq 1$. In the first case it is known that the ABC system is completely integrable, here we provide its explicit first integrals. In the second case Ziglin ["Dichotomy of the separatrices and the nonexistence of first integrals in systems of differential equations of Hamiltonian type with two degrees of freedom," Izv. Akad. Nauk SSSR, Ser. Mat. **51**, 1088 (1987)] proved that the ABC system with $0 < B < 1$ and $C > 0$ sufficiently small has no real meromorphic first integrals. We improve Ziglin's result showing that there are no C^1 first integrals under convenient assumptions. © *2012 American Institute of Physics*. [doi:10.1063/1.3682692]

I. INTRODUCTION

The ABC system was introduced by Arnold for studying the steady state solutions of Euler's hydrodynamic partial differential equations. Here we study the existence and non-existence of first integrals for this differential system defined in the three-dimensional torus. As far as we know we provide by first time sufficient conditions in order that the ABC system has no C^1 first integrals.

The nonlinear ordinary differential equations appear in a natural way in many branches of applied mathematics, physics, chemistry, economy, etc. In particular the ABC system was introduced by Arnold in Ref. 1 and studied in Refs. 5 and 6 with regard to the hydrodynamic instability criterion of Friedlander and Vishik,[4] putting in particular special interest in the existence of a single hyperbolic periodic solution of an ABC system which implies that the associated steady state solution of Euler's equation is hydrodynamically unstable, see also Ref. 3.

Here we are interested in the integrability of the ABC system. More precisely, the goal of this paper is to study the existence of first integrals of the ABC systems

$$\begin{aligned} \dot{x} &= A \sin z + C \cos y, \\ \dot{y} &= B \sin x + A \cos z, \\ \dot{z} &= C \sin y + B \cos x, \end{aligned} \tag{1}$$

in $\mathbb{T}^3$ with the coordinates x, y, z (mod. 2π) and the parameters $A, B, C \in \mathbb{R}$. As usual the dot denotes derivative with respect to time t.

In all the paper we assume that $A \geq 0$, $B \geq 0$, and $C \geq 0$ because doing the change of variables $z \mapsto z + \pi$ we can consider $A \geq 0$, and similarly with the parameters B and C.

If $ABC = 0$, then doing cyclic permutations of x, y, z and of the parameters A, B, C, if necessary, we can consider $A = 0$.

When $ABC \neq 0$ doing a rescaling of time $ds = Mdt$ where $M = \max\{A, B, C\}$ and taking cyclic permutations with respect to the variables x, y, z and the parameters A, B, C, if necessary, we can take $A = 1$ and $0 < C \leq B \leq 1$ or $0 < B < C \leq 1$.

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