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First integrals for a charged particle moving on a plane under the action of a magnetic field orthogonal to this plane

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ABSTRACT

We characterize the polynomial (respectively analytical) first integrals of degree one in the variables u and v of the differential systems of the form $x'=u$, $y'=v$, $u'=B(x,y)v$ and $v'=-B(x,y)u$ where $B=B(x,y)$ is a polynomial (respectively analytic function) in the variables x and y. This differential system models a non-relativistic charge moving on a plane under the action of a magnetic field orthogonal to this plane.

1. Introduction and statement of the main results

One of the more classical problems in the qualitative theory of differential systems is to characterize the existence or not of first integrals.

We consider the differential equation of a non-relativistic charge moving on a plane under the action of a magnetic field orthogonal to this plane. The differential equations of this motion can be written as

$$\ddot{x}=B(x,y)\dot{y}, \quad \ddot{y}=-B(x,y)\dot{x},$$

where B is a non-zero function in the variables x and y. For more details see [2]. The dot denotes derivative with respect to the time t.

We consider the equivalent differential system of first order in $\mathbb{R}^4$

$$\dot{x}=u, \quad \dot{y}=v, \quad \dot{u}=B(x,y)v, \quad \dot{v}=-B(x,y)u. \tag{1}$$

Let $\dot{\mathbf{x}}=f(\mathbf{x})$ be a C^1 differential system in $\mathbb{R}^n$ and let $U\subset\mathbb{R}^n$ be an open set. We say that the non-constant function $H:\mathbb{R}^n\to\mathbb{R}$ is a *first integral* of $\dot{\mathbf{x}}=f(\mathbf{x})$ on U, if $H(x_1(t),\dots,x_n(t))=$ constant for all values of t for which the solution $(x_1(t),\dots,x_n(t))$ of $\dot{\mathbf{x}}=f(\mathbf{x})$ is defined on U.

Let $H_k:U_k\to\mathbb{R}$ for $k=1,\dots,r$ be first integrals of $\dot{\mathbf{x}}=f(\mathbf{x})$. We say that they are *independent* on $U_1\cap\cdots\cap U_r$ if their gradients are linearly independent over a full Lebesgue measure subset of $U_1\cap\cdots\cap U_r$. We say that $\dot{\mathbf{x}}=f(\mathbf{x})$ is *completely integrable* if it has $n-1$ independent first integrals.

It is known that system (1) has the first integral

$$H_1=u^2+v^2.$$

A *polynomial first integral* is a first integral H which is a polynomial. An *analytic first integral* is a first integral H which is an analytic function.

We classify the integrable systems (1) for different classes of first integrals.

Theorem 1. *Let $b(z)$ be a polynomial in the variable z and let $T(x,y)$ be a polynomial in the variables x and y such that $\partial T/\partial x$ and $\partial T/\partial y$ are coprime.*

(a) *System (1) with $B(x,y)=b(T(x,y))$ has a polynomial first integral independent with H_1 of degree one in the variables u and v if and only if*

$$T=a+bx+cy+d(x^2+y^2), \tag{2}$$

with $a,b,c,d\in\mathbb{R}$. Moreover a polynomial first integral independent with H_1 is

$$H=H(x,y,u,v)=\int b(T)\,dT-cu+bv+2d(xv-yu). \tag{3}$$

(b) *If (a) holds then system (1) is completely integrable with analytic first integrals.*

Theorem 2. *Let $b(z)$ be an analytic function in the variable z and let $T(x,y)$ be an analytic function in the variables x and y such that $\partial T/\partial x$ and $\partial T/\partial y$ are coprime.*

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