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On the polynomial integrability of the Hoyer systems

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ABSTRACT

The Hoyer polynomial differential systems depend on nine parameters. We provide the necessary conditions in order that these systems have two functionally independent polynomial first integrals. We show that these conditions are not sufficient. Additionally, we illustrate how the polynomial first integrals of these systems can be computed using the Kowalevsky exponents.

1. Introduction and statement of the main results

Given a system of ordinary differential equations depending on parameters in general it is very difficult to recognize for which values of the parameters the equations have first integrals because there are no satisfactory methods to answer this question.

In this paper we study the polynomial first integrals of the so-called Hoyer systems in $\mathbb{R}^3$ depending on nine parameters. These differential systems were introduced by Hoyer [1] in 1879 in his Ph.D. thesis. They have the form

$$\begin{aligned} \dot{x} &= ayz + bxz + cxy = P_1(x,y,z), \\ \dot{y} &= Ayz + Bxz + Cxy = P_2(x,y,z), \\ \dot{z} &= \alpha yz + \beta xz + \gamma xy = P_3(x,y,z), \end{aligned} \tag{1}$$

where $a, b, c, A, B, C, \alpha, \beta, \gamma \in \mathbb{R}$. These systems are the most general quadratic systems without self-interacting terms.

Among examples of Hoyer systems (1) there are the Euler systems describing the motion of a free rigid body, the (A, B, C) Lotka–Volterra systems and the Halphen systems [2]. Papers of Moulin Ollagnier about polynomial first integrals [3] and rational first integrals of degree zero [4] for the three dimensional homogeneous Lotka–Volterra systems show that searching for those parameter values for which the systems possess a first integral of a specified class is a very hard problem. The Hoyer systems admitting a quadratic polynomial first integral and a Poisson structure have been studied by Maciejewski and Przybylska in [5].

Given U an open set of $\mathbb{R}^3$, we say that a real non-constant function $H: U \to \mathbb{R}$ is a *first integral* if it is constant on every solution of system (1) contained in U, i.e., H satisfies

$$\frac{\partial H}{\partial x}P_1(x,y,z) + \frac{\partial H}{\partial y}P_2(x,y,z) + \frac{\partial H}{\partial z}P_3(x,y,z) = 0$$

on the points on U.

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