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Limit cycles for a generalization of polynomial Liénard differential systems

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ABSTRACT

We study the number of limit cycles of the polynomial differential systems of the form

$$\dot{x} = y - f_1(x)y, \quad \dot{y} = -x - g_2(x) - f_2(x)y,$$

where $f_1(x) = \varepsilon f_{11}(x) + \varepsilon^2 f_{12}(x) + \varepsilon^3 f_{13}(x)$, $g_2(x) = \varepsilon g_{21}(x) + \varepsilon^2 g_{22}(x) + \varepsilon^3 g_{23}(x)$ and $f_2(x) = \varepsilon f_{21}(x) + \varepsilon^2 f_{22}(x) + \varepsilon^3 f_{23}(x)$ where f_{1i}, f_{2i} and g_{2i} have degree l, n and m respectively for each $i = 1,2,3$, and ε is a small parameter. Note that when $f_1(x) = 0$ we obtain the generalized polynomial Liénard differential systems. We provide an accurate upper bound of the maximum number of limit cycles that this differential system can have bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of third order.

1. Introduction

The second part of the 16th Hilbert's problem wants to find an upper bound on the maximum number of limit cycles that the class of all polynomial vector fields with a fixed degree can have. In this paper we will try to give a partial answer to this problem for the class of polynomial differential systems

$$\dot{x} = y - f_1(x)y, \quad \dot{y} = -x - g_2(x) - f_2(x)y, \tag{1}$$

where $f_1(x) = \varepsilon f_{11}(x) + \varepsilon^2 f_{12}(x) + \varepsilon^3 f_{13}(x)$, $g_2(x) = \varepsilon g_{21}(x) + \varepsilon^2 g_{22}(x) + \varepsilon^3 g_{23}(x)$ and $f_2(x) = \varepsilon f_{21}(x) + \varepsilon^2 f_{22}(x) + \varepsilon^3 f_{23}(x)$ where f_{1i}, f_{2i} and g_{2i} have degree l, n and m respectively for each $i = 1,2,3$, and ε is a small parameter. When $f_1(x) = 0$ these systems coincide with the generalized polynomial Liénard differential systems

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y, \tag{2}$$

where $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m, respectively. The classical polynomial Liénard differential systems are

$$\dot{x} = y, \quad \dot{y} = -x - f(x)y, \tag{3}$$

where $f(x)$ is a polynomial in the variable x of degree n. For these systems in 1977 Lins et al. [15] stated the conjecture that if $f(x)$ has degree $n \geqslant 1$ then system (3) has at most $[n/2]$ limit cycles. They prove this conjecture for $n = 1,2$. The conjecture for $n = 3$ has been proved recently by Chengzi and Llibre in [16]. For $n \geqslant 5$ the conjecture is not true, see De Maesschalck and Dumortier [7] and Dumortier et al. [8]. So it remains to know if the conjecture is true or not for $n = 4$.

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singular point (i.e., from a Hopf bifurcation), that are called *small amplitude limit cycles*, see for instance [20]. There are partial results concerning the maximum number of small amplitude limit cycles for Liénard polynomial differential systems. Of course, the number of small amplitude limit cycles gives a lower bound for the maximum number of limit cycles that a polynomial differential system can have.

There are many results concerning the existence of small amplitude limit cycles for the following generalized Liénard polynomial differential system (2). We denote by $H(m,n)$ the number of limit cycles that systems (2) can have. This number is usually called the *Hilbert number* for systems (2).

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