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On the Darboux integrability of Blasius and Falkner–Skan equation

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ABSTRACT

We study the Darboux integrability of the celebrated Falkner–Skan equation $f''' + ff'' + \lambda(1 - f'^2) = 0$, where λ is a parameter. When $\lambda = 0$ this equation is known as Blasius equation. We show that both differential systems have no first integrals of Darboux type. Additionally we compute all the Darboux polynomials and all the exponential factors of these differential equations.

1. Introduction and statement of the main results

The *Falkner–Skan equation* is

$$f''' + ff'' + \lambda(1 - f'^2) = 0, \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter. This equation was first derived in [6] as a model of the steady two-dimensional flow of a slightly viscous incompressible fluid past a wedge. The special case $\lambda = 0$, in which the wedge reduces to a flat plate, is called *Blasius equation* and was considered for a first time in [2].

Both equations are the subject of an extensive literature. For the derivation of this equation see [1]. For the existence and uniqueness of the solutions see, for example, [19,22,5,18,3,13] and references therein. Recently there has been also a renewed interest in the mathematical aspects of the Falkner–Skan equation. The dynamic features of this equation such as the existence of oscillating and periodic orbits have been studied in [10–12,16]; and for more recent works on the bifurcations in this equation see [14,21,20].

In MathSciNet appears in this moment 214 articles related with the Falkner–Skan equation, but any of these papers analyze the integrability or non-integrability of this equation. In this work we are interested in the Darboux integrability of Blasius and Falkner–Skan equation. Before we state our main result (Theorem 1) we need to introduce some definitions and auxiliary results.

We can express (1) as a system of differential equations

$$\dot{x} = y, \quad \dot{y} = z \quad \dot{z} = -xz - \lambda(1 - y^2), \tag{2}$$

and the associated vector field is

$$\mathcal{X} = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + [-xz - \lambda(1 - y^2)]\frac{\partial}{\partial z}. \tag{3}$$

Let $U \subset \mathbb{R}^3$ be an open subset. We say that the non-constant function $H : U \to \mathbb{R}$ is the *first integral* of the polynomial vector field (3) on U associated to system (2), if $H(x(t), y(t), z(t))$ = constant for all values of t for which the solution $(x(t), y(t), z(t))$ of $\mathcal{X}$ is defined on U. Clearly H is a first integral of $\mathcal{X}$ on U if and only if $\mathcal{X}H = 0$ on U. When H is a polynomial we say that H is a *polynomial first integral*.

For proving our main results concerning the existence of first integrals of Darboux type we shall use the invariant algebraic surfaces of system (2). This is the basis of the Darboux theory of integrability. The Darboux theory of integrability works for real or complex polynomial ordinary differential equations. The study of complex invariant algebraic curves is necessary for obtaining all the real first integrals of a real polynomial differential equation, for more details see [7–9,15,17].

Let $h = h(x, y, z) \in \mathbb{C}[x, y, z]$ be a non-constant polynomial. We say that $h = 0$ is an *invariant algebraic surface* of the vector field $\mathcal{X}$ in (3) if it satisfies $\mathcal{X}h = Kh$, for some polynomial $K = K(x, y, z) \in \mathbb{C}[x, y, z]$, called the *cofactor* of h. Note that K has degree at most 1. The polynomial h is called a *Darboux polynomial*, and we also say that K is the *cofactor* of the Darboux polynomial h. We note that a Darboux polynomial with zero cofactor is a polynomial first integral.

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