WEIERSTRASS INTEGRABILITY FOR THE DIFFERENTIAL SYSTEMS $x' = y, \quad y' = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x)$

JAUME LLIBRE[1] AND CLÀUDIA VALLS[2]

Abstract. We characterize the differential equations of the form

$$x' = y, \quad y' = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x), \quad n \geq 2, \quad a_n(0) \neq 0,$$

where $a_j(x)$ are meromorphic functions in the variable x for $j = 0, \ldots, n$ that admits either a Weierstrass first integral or a Weierstrass inverse integrating factor.

1. Introduction and statement of the main results

Let x and y be complex variables. In this paper we study the differential equations of the form

$$x' = y, \quad y' = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x), \quad n \geq 2, \quad a_n(0) \neq 0, \tag{1}$$

where $a_j(x)$ are meromorphic functions in the variable x for $j = 0, \ldots, n$ with $n \geq 2$ and the prime indicates derivative with respect to the time t, real or complex.

The goal of this paper is to analyze the integrability of the differential systems (1) restricted to a special kind of first integrals. For such systems the notion of integrability is based on the existence of a first integral, and we shall characterize when the differential system (1) has a Weierstrass first integral or a Weierstrass inverse integrating factor. More precisely, guided by the fact that system (1) is polynomial in the variable y, we study the first integrals and the inverse integrating factors that are polynomials in the variable y and are an analytic function in the variable x, i.e. we study the so called Weierstrass integrability of system (1).

As usual $\mathbb{C}[[x]]$ is the ring of formal power series in the variable x with coefficients in $\mathbb{C}$, and $\mathbb{C}[y]$ is the ring of polynomials in the variable y with coefficients in $\mathbb{C}$. A polynomial of the form

$$\sum_{i=0}^{n} w_i(x)y^i \in \mathbb{C}[[x]][y], \tag{2}$$

is called a *formal Weierstrass polynomial* in the variable y of degree n if and only if $w_n(x) = 1$ and $w_i(0) = 0$ for $i < n$. A formal polynomial whose coefficients are convergent is called *Weierstrass polynomial*, see [1]. In other words, a *Weierstrass polynomial first integral* is of the form

$$H = y^s + H_{s-1}(x)y^{s-1} + \cdots + H_1(x)y + H_0(x) = \sum_{i=0}^{s} H_i(x)y^i \tag{3}$$

2010 *Mathematics Subject Classification.* Primary 34C05, 34A34, 34C14.

Key words and phrases. Weierstrass first integrals, Weierstrass inverse integrating factor, complex differential systems.