POLYNOMIAL FIRST INTEGRALS FOR WEIGHT-HOMOGENEOUS PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS OF WEIGHT DEGREE 4

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Abstract. We classify all the weight-homogeneous planar polynomial differential systems of weight degree 4 having a polynomial first integral.

1. Introduction and statement of the main result

In this paper we deal with polynomial differential systems of the form

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \qquad \mathbf{x} = (x, y) \in \mathbb{C}^2, \tag{1}$$

with $\mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), P_2(\mathbf{x}))$ and $P_i \in \mathbb{C}[x, y]$ for $i = 1, 2$. As usual $\mathbb{Q}^+$, $\mathbb{R}$ and $\mathbb{C}$ will denote the sets of non–negative rational, real and complex numbers, respectively; and $\mathbb{C}[x, y]$ denotes the polynomial ring over $\mathbb{C}$ in the variables x, y. Here, t is real or complex.

We say that system (1) is *weight homogeneous* or *quasi–homogeneous* if there exist $\mathbf{s} = (s_1, s_2) \in \mathbb{N}^2$ and $d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R},\ a > 0\}$,

$$P_i(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^{s_i - 1 + d} P_i(x, y), \tag{2}$$

for $i = 1, 2$. We call $\mathbf{s} = (s_1, s_2)$ the *weight exponent* of system (1), and d the *weight degree* with respect to the weight exponent $\mathbf{s}$. In the particular case that $\mathbf{s} = (1, 1)$ system (1) is called a *homogeneous polynomial differential system of degree* d.

Recently such systems have been investigated by several authors. Labrunie [12] and Moulin Ollagnier [15] characterize all polynomial first integrals of the three dimensional (a, b, c) Lotka–Volterra systems. Maciejewski and Strelcyn [14] proved that the so–called Halphen system has no algebraic first integrals. But some of the best results for general weight homogeneous polynomial differential systems have been provided by Furta [10] and Goriely [11], and additionally for quadratic homogeneous polynomial differential systems by Tsygvintsev [16], and Llibre and Zhang [13]. See also the works of Algaba, Freire, Fuentes, Garcia and Teixeira [1]–[5].

A non–constant function $H(x, y)$ is a *first integral* of system (1) if it is constant on all solution curves $(x(t), y(t))$ of system (1); i.e. $H(x(t), y(t)) =$

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constant for all values of t for which the solution $(x(t), y(t))$ is defined. If H is C^1, then H is a first integral of system (1) if and only if

$$P_1 \frac{\partial H}{\partial x} + P_2 \frac{\partial H}{\partial y} = 0. \tag{3}$$

The function $H(x, y)$ is *weight homogeneous of weight degree* m *with respect to the weight exponent* $\mathbf{s}$ if it satisfies $H(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^m H(x, y)$, for all $\alpha \in \mathbb{R}^+$.

Given $H \in \mathbb{C}[x, y]$ we can split it into the form $H = H_m + H_{m+1} + \ldots + H_{m+l}$, where H_{m+i} is a *weight homogeneous* polynomial of weight degree $m+i$ with respect to the weight exponent $\mathbf{s}$; i.e. $H_{m+i}(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^{m+i} H_{m+i}(x, y)$. The following well-known proposition (see [13] for a proof) reduces the study of the existence of analytic first integrals of a weight-homogeneous polynomial differential system (1) to the study of the existence of a weight-homogeneous polynomial first integral.

Proposition 1. *Let* H *be an analytic function and let* $H = \sum_i H_i$ *be its decomposition into weight-homogeneous polynomials of weight degree* i *with respect to the weight exponent* $\mathbf{s}$. *Then* H *is an analytic first integral of the weight-homogeneous polynomial differential system* (1) *if and only if each weight-homogeneous part* H_i *is a first integral of system* (1) *for all* i.

The main goal of this paper is to classify all analytic first integrals of the weight-homogenous planar polynomial differential systems of weight degree 4. In view of Proposition 1 we only need to classify all the polynomial first integrals of the weight-homogenous planar polynomial differential systems of weight degree 4. The classification of all polynomial first integrals (and hence of all analytic first integrals) of the weight-homogenous planar polynomial differential systems of weight degree 1 is straightforward and trivial. The classification of all polynomial first integrals (and hence of all analytic first integrals) of the weight-homogenous planar polynomial differential systems of weight degree 2 was given in [13, 8] and for systems of weight degree 3 was given in [6, 8].

In the classification of all polynomial first integrals for weight-homogenous planar polynomial differential systems of weight degree 2 and 3 the authors use the Kowalevskaya exponents, but as it was shown in Theorems 4 of [6] these exponents are useless for classifying the polynomial first integrals for weight-homogenous planar polynomial differential systems of weight degree larger than 3.

Proposition 2. *In $\mathbb{C}^2$ the systems with weight degree 4 can be written as the following ones with their corresponding values of* $\mathbf{s}$*:*

$$
\begin{aligned}
\mathbf{s}=(1,1):\quad & \dot{x}=a_{40}x^4+a_{31}x^3y+a_{22}x^2y^2+a_{13}xy^3+a_{04}y^4,\\
& \dot{y}=b_{40}x^4+b_{31}x^3y+b_{22}x^2y^2+b_{13}xy^3+b_{04}y^4;\\
\mathbf{s}=(1,2):\quad & \dot{x}=a_{40}x^4+a_{21}x^2y+a_{02}y^2,\\
& \dot{y}=b_{50}x^5+b_{31}x^3y+b_{12}xy^2;\\
\mathbf{s}=(1,3):\quad & \dot{x}=a_{40}x^4+a_{11}xy,\\
& \dot{y}=b_{60}x^6+b_{31}x^3y+b_{02}y^2;\\
\mathbf{s}=(1,4):\quad & \dot{x}=a_{40}x^4+a_{01}y,\\
& \dot{y}=b_{70}x^7+b_{31}x^3y;\\
\mathbf{s}=(2,3):\quad & \dot{x}=a_{11}xy\\
& \dot{y}=b_{30}x^3+b_{02}y^2;\\
\mathbf{s}=(2,5):\quad & \dot{x}=a_{01}y,\\
& \dot{y}=b_{40}x^4;\\
\mathbf{s}=(3,3):\quad & \dot{x}=a_{20}x^2+a_{11}xy+a_{02}y^2,\\
& \dot{y}=b_{20}x^2+b_{11}xy+b_{02}y^2;\\
\mathbf{s}=(6,9):\quad & \dot{x}=a_{01}y,\\
& \dot{y}=b_{20}x^2.
\end{aligned}
$$

Proposition 2 is proved in section 2.

In what follows we state our main results, i.e. we classify when the systems of Proposition 2 exhibit a polynomial first integral. The systems with weight exponent $(1,1)$ having a polynomial first integral are given in section 3, because its classification is very long. The systems with weight exponent $(3,3)$ having a polynomial first integral are studied inside the systems with weight exponent $(1,2)$. For the other systems of Proposition 2 we provide in this introduction their polynomial first integrals.

We introduce the change $(X,Y)=(x^2,y)$ in the planar weight homogeneous polynomial differential systems (1) of weight degree 4 with weight exponent $(1,2)$. With these new variables (X,Y) the systems with weight exponent $(1,2)$ becomes, after introducing the new independent variable $d\tau = x\,dt$,

$$X' = 2a_{40}X^2 + 2a_{21}XY + 2a_{02}Y^2, \quad Y' = b_{50}X^2 + b_{31}XY + b_{12}Y^2, \tag{4}$$

where the prime denotes derivative with respect to τ.

System (4) is a homogeneous quadratic planar polynomial system (with $\mathbf{s}=(1,1)$). It is well-known, see [9], that for each quadratic homogeneous system there exists some linear transformation and a rescaling of time which transforms system (4) into systems in (5).

$$
\begin{array}{ll}
\dot{x} = -2xy + \frac{2}{3}x(p_1x + p_2y), & \dot{y} = -x^2 + y^2 + \frac{2}{3}y(p_1x + p_2y), \\
\dot{x} = -2xy + \frac{2}{3}x(p_1x + p_2y), & \dot{y} = x^2 + y^2 + \frac{2}{3}y(p_1x + p_2y), \\
\dot{x} = -x^2 + \frac{2}{3}x(p_1x + p_2y), & \dot{y} = 2xy + \frac{2}{3}y(p_1x + p_2y), \\
\dot{x} = \frac{2}{3}x(p_1x + p_2y), & \dot{y} = x^2 + \frac{2}{3}y(p_1x + p_2y), \\
\dot{x} = \frac{2}{3}x(p_1x + p_2y), & \dot{y} = \frac{2}{3}y(p_1x + p_2y).
\end{array} \tag{5}
$$

We prove the following theorem that characterizes all the polynomial first integrals for the systems in (5).

Theorem 1. *The homogeneous polynomial systems in* (5) *have a polynomial first integral H if and only if one of the following conditions hold.*

(a) *The first system in* (5) *with $p_1 = 0$, $p_2 = 3(1-q)/(1+2q)$ with $q = n/m \in \mathbb{Q}^+$, and in this case $H = x^m(3y^2 - x^2)^n$.*
(b) *The second system in* (5) *with $p_1 = 0$, $p_2 = 3(1-q)/(1+2q)$ with $q = n/m \in \mathbb{Q}^+$, and in this case $H = x^m(3y^2 + x^2)^n$.*
(c) *The third system in* (5) *with $p_1 = 0$ and $p_2 = 3(1-2q)/(2(1+q))$ with $q = n/m \in \mathbb{Q}^+$, and in this case $H = x^m y^n$.*
(d) *The fourth system in* (5) *with $p_1 = p_2 = 0$, and in this case $H = x$.*

We note that systems with weight exponent $(3,3)$ coincide with the systems (4) and hence it can be written into systems in (5). Therefore, Theorem 1 applies to those systems. The proof of Theorem 1 is given in section 4.

Theorem 2. *The weight homogeneous polynomial differential systems with weight exponent $(1,3)$ and weight degree 4 have a polynomial first integral H if and only if the following conditions hold.*

(a) *$a_{11} = a_{40} = 0$ with $H = x$.*
(b) *$b_{60} = b_{31} = b_{02} = 0$ with $H = y$.*
(c) *$(3a_{11}-b_{02})(3a_{11}-2b_{02}) \neq 0$, $a_{40} = -a_{11}b_{31}/(3a_{11}-2b_{02})$, $3a_{11}/(6a_{11}-2b_{02}) = m/n \in \mathbb{Q}^+$ and $m/n < 1$ with*

$$H = x^{3(n-m)}((3a_{11}-2b_{02})b_{60}x^6 + 2(3a_{11}-b_{02})b_{31}x^3y - (9a_{11}^2 - 9a_{11}b_{02} + 2b_{02}^2)y^2)^m.$$

(d) *$b_{31}/a_{40} = -m/n$ and $m/n \in \mathbb{Q}^+$ with $H = x^{3m}(b_{60}x^3 + (b_{31} - 3a_{40})y)^{3n}$.*
(e) *$b_{02} = 0$, $a_{11} \neq 0$, $a_{40} = -b_{31}/3$ and $b_{06} = -(3a_{40} - b_{31})^2/(12a_{11})$ with $H = b_{31}x^3 - 3a_{11}y$.*
(f) *$(3a_{11} - b_{02})(3a_{11} - 2b_{02}) \neq 0$, $a_{40} = -a_{11}b_{31}/(3a_{11} - 2b_{02})$, $b_{06} = -(3a_{40} - b_{31})^2/(4(3a_{11} - b_{02}))$, $b_{02} \neq 0$ and $-3a_{11}/b_{02} = n/m \in \mathbb{Q}^+$ with*

$$H = x^{3n}(b_{31}x^3 + (2b_{02} - 3a_{11})y)^m.$$

The proof of Theorem 2 is given in section 5.

Theorem 3. *The weight homogeneous polynomial differential systems with weight exponent $(1,4)$ and weight degree 4 have a polynomial first integral H if and only if the following conditions hold.*

(a) *$a_{40} = a_{01} = 0$ and $H = x$.*
(b) *$b_{70} = b_{31} = 0$ and $H = y$.*

(c) $b_{31} = -4a_{40}$, *and* $4a_{40}^2 + a_{01}b_{70} \neq 0$ *with* $H = -b_{70}x^8 + 8a_{40}x^4y + 4a_{01}y^2$.

(d) $b_{31} = -4a_{40}$, $a_{40}b_{70} \neq 0$ *and* $4a_{40}^2 + a_{01}b_{70} = 0$ *with* $H = b_{70}x^4 - 4a_{40}y$.

Theorem 3 is proved in section 6.

Theorem 4. *The weight homogeneous polynomial differential systems with weight exponent* $(2, 3)$ *and weight degree* 4 *have a polynomial first integral* H *if and only if* $a_{11} = 0$ *in which case* $H = x$, *or* $b_{30} = b_{02} = 0$ *in which case* $H = y$, *or* $a_{11}(3a_{11} - 2b_{02}) \neq 0$ *and* $-2b_{02}/a_{11} = n/m \in \mathbb{Q}^+$, *in which case*

$$H = x^n(2b_{30}x^3 - 3a_{11}y^2 + 2b_{02}y^2)^m.$$

The proof of Theorem 4 is given in section 7.

Theorem 5. *The weight homogeneous polynomial differential systems with weight exponent* $(2, 5)$ *and weight degree* 4 *have the polynomial first integral* $H = 2b_{40}x^5 - 5a_{01}y^2$.

The proof of Theorem 5 is given in section 8.

Theorem 6. *The weight homogeneous polynomial differential systems with weight exponent* $(6, 9)$ *and weight degree* 4 *have the polynomial first integral* $H = 2b_{20}x^3 - 3a_{01}y^2$.

The proof of Theorem 6 is given in section 9.

2. Proof of Proposition 2

From the definition of weight homogeneous polynomial differential systems (1) with weight degree 4, the exponents u_i and v_i of any monomial $x^{u_i}y^{v_i}$ of P_i for $i = 1, 2$, are such that they satisfy respectively the relations

$$s_1u_1 + s_2v_1 = s_1 + 3, \quad \text{and} \quad s_1u_2 + s_2v_2 = s_2 + 3. \tag{6}$$

Moreover, we can assume that P_1 and P_2 are coprime, and without loss of generality we can also assume that $s_1 \leq s_2$. We consider different values of s_1.

Case $s_1 = 1$. If $s_2 = 1$ then in view of (6) we must have $u_1 + v_1 = 4$ and $u_2 + v_2 = 4$, that is, $(u_i, v_i) = (0, 4)$, $(u_i, v_i) = (1, 3)$, $(u_i, v_i) = (2, 2)$, $(u_i, v_i) = (3, 1)$ and $(u_i, v_i) = (4, 0)$ for $i = 1, 2$.

If $s_2 = 2$ then in view of (6) we must have $u_1 + 2v_1 = 4$ and $u_2 + 2v_2 = 5$, that is, $(u_1, v_1) = (0, 2)$, $(u_1, v_1) = (2, 1)$ and $(u_1, v_1) = (4, 0)$, while $(u_2, v_2) = (1, 2)$, $(u_2, v_2) = (3, 1)$ and finally $(u_2, v_2) = (5, 0)$.

If $s_2 = 3$ then in view of (6) we must have $u_1 + 3v_1 = 4$ and $u_2 + 3v_2 = 6$, that is, $(u_1, v_1) = (1, 1)$, $(u_1, v_1) = (4, 0)$, while $(u_2, v_2) = (0, 2)$, $(u_2, v_2) = (3, 1)$ and finally $(u_2, v_2) = (6, 0)$.

If $s_2 = 4$ then in view of (6) we must have $u_1 + 4v_1 = 4$ and $u_2 + 4v_2 = 7$, that is, $(u_1, v_1) = (0, 1)$, $(u_1, v_1) = (4, 0)$, while $(u_2, v_2) = (3, 1)$ and finally $(u_2, v_2) = (7, 0)$.

If $s_2 = 4 + l$ with $l \geq 1$, then equation (6) becomes

$$u_1 + (4 + l)v_1 = 4 \quad \text{and} \quad u_2 + (4 + l)v_2 = 7 + l. \tag{7}$$

From the first equation of (7) we get $v_1 = 0$ and $u_1 = 4$. By the second equation of (7) it follows that $v_2 \in \{0, 1\}$. If $v_2 = 0$ then $u_2 = 7 + l$, and if $v_2 = 1$ then $u_2 = 3$. In both cases P_1 and P_2 are not coprime. So this case is not considered.

Case $s_1 = 2$. Now we have $s_2 \geq 2$. If $s_2 = 2$ then in view of (6) we must have $2u_1 + 2v_1 = 5$ and $2u_2 + 2v_2 = 5$, which is not possible because 5 is not an even number.

If $s_2 = 3$ then in view of (6) we must have $2u_1 + 3v_1 = 5$ and $2u_2 + 3v_2 = 6$, that is, $(u_1, v_1) = (1, 1)$, while $(u_2, v_2) = (0, 2)$ and $(u_2, v_2) = (3, 0)$.

If $s_2 = 4$ then in view of (6) we must have $2u_1 + 4v_1 = 5$ and $2u_2 + 4v_2 = 7$, which is not possible because 5 is not even.

If $s_2 = 5$ then in view of (6) we must have $2u_1 + 5v_1 = 5$ and $2u_2 + 5v_2 = 8$, that is, $(u_1, v_1) = (0, 1)$ and $(u_2, v_2) = (4, 0)$.

If $s_2 = 5 + l$ with $l \geq 1$, then equation (6) becomes

$$2u_1 + (5 + l)v_1 = 5 \quad \text{and} \quad 2u_2 + (5 + l)v_2 = 8 + l. \tag{8}$$

The first equation of (8) is not possible because 5 is not an even number, $5 + l \geq 6$ and u_1, v_1 are non-negative integers.

Case $s_1 = 3$. Now we have $s_2 \geq 3$. If $s_2 = 3$ then in view of (6) we must have $3u_1 + 3v_1 = 6$ and $3u_2 + 3v_2 = 6$, that is, $(u_i, v_i) = (0, 2)$, $(u_i, v_i) = (1, 1)$, $(u_i, v_i) = (2, 0)$, for $i = 1, 2$.

If $s_2 = 3 + l$ with $l \geq 1$. In view of (6) we must have

$$3u_1 + (3 + l)v_1 = 6 \quad \text{and} \quad 3u_2 + (3 + l)v_2 = 6 + l. \tag{9}$$

From the first equation of (9) we have that

$$v_1 = \frac{6 - 3u_1}{3 + l} \leq \frac{6}{3 + l},$$

and using that $l \geq 1$ then $v_1 \in \{0, 1\}$.

When $v_1 = 0$ then $3u_1 = 6$ and thus $u_1 = 2$. Then from the second equation of (9) we get that $v_2 \in \{0, 1\}$. If $v_2 = 0$ then $u_2 \geq 1$, and if $v_2 = 1$ then $u_2 = 1$. In both cases we have that P_1 and P_2 are not coprime.

When $v_1 = 1$ then $3u_1 = 3 - l$, which is not possible since u_1 is an integer and $l \geq 1$.

Case $s_1 = 3 + l$ *with* $l \geq 1$. Now we have $s_2 \geq 3 + l$ with $l \geq 1$ and equation (6) becomes

$$(3 + l)u_1 + s_2v_1 = 6 + l = (3 + l) + 3 \quad \text{and} \quad (3 + l)u_2 + s_2v_2 = 3 + s_2. \tag{10}$$

From the first equation of (10) and taking into account that $l \geq 1$, we get that $u_1 \in \{0, 1\}$.

When $u_1 = 0$ we must have $s_2v_1 = 6 + l$, and since $s_2 \geq 3 + l$ we get

$$v_1 = \frac{6 + l}{s_2} \leq \frac{(3 + l) + 3}{3 + l} = 1 + \frac{3}{3 + l}.$$

Since $v_1 \neq 0$ and $l \geq 1$ we must have $v_1 = 1$, and then $s_2 = 6 + l$. Now the second equation of (10) becomes

$$(3 + l)u_2 + (6 + l)v_2 = (6 + l) + 3. \tag{11}$$

Then $v_2 = 0$. From (11) we have $u_2 = 1 + \dfrac{6}{l+3}$. Thus $l = 3$ and $u_2 = 2$. So we get the systems with weight exponent $(6, 9)$.

When $u_1 = 1$ we must have $s_2 v_1 = 3$ and since $s_2 \geq 3 + l$ we get $(3+l)v_2 \leq 3$, which is not possible because $l \geq 1$. This concludes the proof of the proposition.

3. Weight exponent $\mathbf{s} = (1, 1)$

A weight homogeneous polynomial system

$$\dot{x} = P_1(x, y); \quad \dot{y} = P_2(x, y),$$

with weight exponent $(1, 1)$ and weight degree d is integrable and its inverse integrating factor is $V(x, y) = xP_2(x, y) - yP_1(x, y)$. See [7] for more details.

As $P_1(x, y), P_2(x, y)$ and $V(x, y)$ are homogeneous polynomials, if the degree of $P_1(x, y)$ and $P_2(x, y)$ is d, then of course, the degree of $V(x, y)$ is $d+1$. Thus, for $d = 4$ we can write the homogeneous polynomials as follows:

$$\begin{aligned} P_1(x,y) = \;& (p_1 - a_1)x^4 + (p_2 - 4a_2)x^3y + (p_3 - 6a_3)x^2y^2 + \\ & (p_4 - 4a_4)xy^3 - a_5y^4, \\ P_2(x,y) = \;& a_0x^4 + (4a_1 + p_1)x^3y + (6a_2 + p_2)x^2y^2 + \\ & (4a_3 + p_3)xy^3 + (a_4 + p_4)y^4, \end{aligned} \tag{12}$$

and

$$V(x, y) = a_0x^5 + 5a_1x^4y + 10a_2x^3y^2 + 10a_3x^2y^3 + 5a_4xy^4 + a_5y^5.$$

So, the first integral is $H(x, y) = \int (P_1(x, y)/V(x, y))dy + g(x)$, satisfying $\partial H/\partial x = -P_2/V$. The canonical forms appear in the factorization of V. Assume that $V(x, y)$ factorizes as

1) 5 simple real roots: $a_0(x - r_1y)(x - r_2y)(x - r_3y)(x - r_4y)(x - r_5y)$,
2) 1 double and 3 simple real roots: $a_0(x-r_1y)^2(x-r_2y)(x-r_3y)(x-r_4y)$,
3) 2 double roots and 1 simple real roots: $a_0(x-r_1y)^2(x-r_2y)^2(x-r_3y)$,
4) 1 triple and 2 simple real roots: $a_0(x - r_1y)^3(x - r_2y)(x - r_3y)$,
5) 1 triple and 1 double real roots: $a_0(x - r_1y)^3(x - r_2y)^2$,
6) 1 quadruple and 1 simple real roots: $a_0(x - r_1y)^4(x - r_2y)$,
7) 1 quintuple real root: $a_0(x - ry)^5$,
8) 3 real and 1 couple of conjugate complex roots: $a_0(x-r_1y)(x-r_2y)(x-r_3y)(x^2 + bxy + cy^2)$ with $b^2 - 4c < 0$,
9) 1 double, 1 simple real and 1 couple of conjugate complex roots: $a_0(x - r_1y)^2(x - r_2y)(x^2 + bxy + cy^2)$ with $b^2 - 4c < 0$,
10) 1 triple real and 1 couple of conjugate complex roots: $a_0(x-r_1y)^3(x^2 + bxy + cy^2)$ with $b^2 - 4c < 0$,
11) 1 simple real and 2 couples of conjugate complex roots: $a_0(x-ry)(x^2 + b_1xy + c_1y^2)(x^2 + b_2xy + c_2y^2)$ with $b_1^2 - 4c_1 < 0, b_2^2 - 4c_2 < 0$,
12) 1 simple real and 1 double couple of conjugate complex roots: $a_0(x - ry)(x^2 + bxy + cy^2)^2$ with $b^2 - 4c < 0$.

Now we shall compute for each case the first integral and obtain the conditions in order that it is a polynomial.

We define the function

$$f(r) = 5(p_4 + p_3r + p_2r^2 + p_1r^3).$$

Case 1): A first integral H is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x-r_3y)^{\gamma_3}(x-r_4y)^{\gamma_4}(x-r_5y)^{\gamma_5},$$

where

$$\gamma_i=\frac{f(r_i)+a_0\prod_{j=1,\ j\neq i}^5(r_i-r_j)}{\prod_{j=1,\ j\neq i}^5(r_i-r_j)}.$$

We note that an integer power of H is a polynomial if and only if $\gamma_i\in\mathbb{Q}$ for $i=1,2,3,4,5$ and they all have the same sign.

Case 2): A first integral H is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x-r_3y)^{\gamma_3}(x-r_4y)^{\gamma_4}$$
$$\exp\left(\frac{f(r_1)x}{r_1(r_1-r_2)(r_1-r_3)(r_1-r_4)(x-r_1y)}\right)$$

with $\gamma_1=A_1/B_1=A_1/[(r_1-r_2)^2(r_1-r_3)^2(r_1-r_4)^2]$ and

$$\begin{aligned}A_1= \ & 5p_1\left((r_2+r_3+r_4)r_1^2-2(r_3r_4+r_2(r_3+r_4))r_1+3r_2r_3r_4\right)r_1^2+\\ & 5p_2\left(r_1^3-(r_3r_4+r_2(r_3+r_4))r_1+2r_2r_3r_4\right)r_1+\\ & 5p_3\left(2r_1^3-(r_2+r_3+r_4)r_1^2+r_2r_3r_4\right)+\\ & 5p_4\left(3r_1^2-2(r_2+r_3+r_4)r_1+r_3r_4+r_2(r_3+r_4)\right)-2a_0B_1,\end{aligned}$$

while for $i=2,\dots,4$ and

$$\gamma_i=\frac{A_i}{B_i}=\frac{-(f(r_i)+a_0B_i)}{(r_1-r_i)^2\prod_{j=2;j\neq i}^4(r_i-r_j)}.$$

We note that an integer power of H is a polynomial if and only if $f(r_1)=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2,3,4$ and they all have the same sign.

Case 3): A first integral H is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x-r_3y)^{\gamma_3}$$
$$\exp\left(-\sum_{i=1}^2\frac{f(r_i)x}{r_i(r_1-r_2)^2(r_i-r_3)(r_iy-x)}\right),$$

with $\gamma_i=A_i/B_i=A_i/[(r_1-r_2)^3(r_i-r_3)^2]$ for $i=1,2$, $\gamma_3=A_3/B_3=-(f(r_3)+a_0B_3)/[(r_1-r_3)^2(r_2-r_3)^2]$,

$$\begin{aligned}A_i= \ & -2a_0B_i+(-1)^{i+1}(5p_4(3r_i-r_j-2r_3)-5p_3((r_1+r_2)r_3-2r_i^2)+\\ & 5p_2r_i(r_i(r_1+r_2)-2r_jr_3)+5p_1r_i^2(-3r_jr_3+r_i(2r_j+r_3))),\end{aligned}$$

for $i,j=1,2$ and $i\neq j$. We note that an integer power of H is a polynomial if and only if $f(r_i)=0$ for $i=1,2$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2,3$ and they all have the same sign.

Case 4): A first integral H is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x-r_3y)^{\gamma_3}$$
$$\exp\left(\frac{5\beta x}{2r_1^2(r_1-r_2)^2(r_1-r_3)^2(r_1y-x)^2}\right),$$

with

$$\begin{aligned}\beta = \; & \big(p_1\left(r_1^2-3r_2r_1-3r_3r_1+5r_2r_3\right)r_1^3-p_2\left(r_1^2+r_2r_1+r_3r_1-3r_2r_3\right)r_1^2\\ & -p_3\left(3r_1^2-r_2r_1-r_3r_1-r_2r_3\right)r_1+p_4\left(-5r_1^2+3r_2r_1+3r_3r_1-r_2r_3\right)\big)x\\ & +2r_1\big(p_1(r_1r_2-2r_3r_2+r_1r_3)r_1^3+p_3(2r_1-r_2-r_3)r_1^2\\ & +p_2\left(r_1^2-r_2r_3\right)r_1^2+p_4\left(3r_1^2-2r_2r_1-2r_3r_1+r_2r_3\right)\big)y,\end{aligned}$$

$\gamma_1 = A_1/B_1 = A_1/[(r_1-r_2)^3(r_1-r_3)^3)]$,

$$\begin{aligned}A_1 = \; & -3a_0B_1-5\big(r_2(p_2+p_1r_2)r_1^3+(r_1-3r_2)(p_2+p_1r_2)r_3r_1^2\\ & +\left(p_2r_2^2+p_1r_1\left(r_1^2-3r_2r_1+3r_2^2\right)\right)r_3^2\big)\\ & +5p_3\left(r_1^3-3r_2r_3r_1+r_2r_3(r_2+r_3)\right)\\ & +5p_4\left(3r_1^2-3(r_2+r_3)r_1+r_2^2+r_3^2+r_2r_3\right),\end{aligned}$$

$\gamma_i = A_i/B_i = (-1)^i(f(r_i)+a_0B_i)/[(r_1-r_i)^3(r_2-r_3)]$ for $i=2,3$. We note that an integer power of H is a polynomial if and only if $\beta=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2,3$ and they all have the same sign.

Case 5): A first integral H is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}\exp(\beta),$$

where

$$\begin{aligned}\beta = \; & \frac{2(r_1-r_2)xf(r_2)r_1^2}{r_2(r_2y-x)}+\frac{(r_1-r_2)^2x^2f(r_1)}{(x-r_1y)^2}+\\ & \frac{10(r_1-r_2)\left((2p_3+2p_1r_1r_2+p_2(r_1+r_2))r_1^2+p_4(3r_1-r_2)\right)x}{r_1y-x},\end{aligned}$$

and

$$\begin{aligned}\gamma_1 = \; & -2r_1^2\big(3a_0(r_1-r_2)^4+15p_4+5p_3(r_1+2r_2)\\ & +5r_2(3p_1r_1r_2+p_2(2r_1+r_2))\big),\\ \gamma_2 = \; & -2r_1^2\big(2a_0(r_1-r_2)^4-15p_4-5p_3(r_1+2r_2)\\ & -5r_2(3p_1r_1r_2+p_2(2r_1+r_2))\big).\end{aligned}$$

We note that an integer power of H is a polynomial if and only if $\beta=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2$ and they all have the same sign.

Case 6): A first integral H is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}\exp(\beta),$$

where

$$\begin{aligned}\beta = \; & \frac{2(r_1-r_2)^3f(r_1)x^3}{r_1^3(r_1y-x)^3}+\\ & \frac{30(r_1-r_2)\left((p_3+r_2(p_2+p_1r_2))r_1^3+p_4\left(3r_1^2-3r_2r_1+r_2^2\right)\right)x}{r_1^3(r_1y-x)}+\\ & \frac{15(r_1-r_2)^2\left(p_4(3r_1-2r_2)+r_1\left((p_2+p_1r_2)r_1^2+p_3(2r_1-r_2)\right)\right)x^2}{r_1^3(x-r_1y)^2},\end{aligned}$$

and

$$\begin{aligned}\gamma_1 = \; & -6\left(-4a_0(r_1-r_2)^4+5p_4+5r_2(p_3+r_2(p_2+p_1r_2))\right),\\ \gamma_2 = \; & 6\left(a_0(r_1-r_2)^4+5p_4+5r_2(p_3+r_2(p_2+p_1r_2))\right).\end{aligned}$$

We note that an integer power of H is a polynomial if and only if $\beta=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2$ and they all have the same sign.

Case 7): A first integral H is

$$(x-ry)^{\gamma_1}\exp\left(\frac{\beta x}{(x-ry)^4}\right),$$

where

$$\begin{aligned}\beta = \;& rx\left(rx(-p_2x+3p_1rx+4p_2ry)+p_3\left(x^2-4ryx+6r^2y^2\right)\right)\\ & -3p_4(x-2ry)\left(x^2-2ryx+2r^2y^2\right),\end{aligned}$$

and $\gamma_1=-12a_0r^4$. We note that $x-ry$ is a polynomial first integral if and only if $\beta=0$.

Case 8): A first integral H is

$$\begin{aligned}&(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x-r_3y)^{\gamma_3}(x^2+bxy+cy^2)^{\gamma_4}\\ &\exp\left(\frac{\beta x}{\prod_{i=1}^3(c+b\,r_i+r_i^2)\sqrt{(4c-b^2)x^2}}\arctan\left(\frac{bx+2cy}{\sqrt{(4c-b^2)\,x^2}}\right)\right),\end{aligned}$$

where

$$\begin{aligned}\beta = \;& 5\big(2p_1c^3-(b(p_2-p_1(r_1+r_2+r_3))+2(p_3+p_2(r_1+r_2+r_3)+\\ & p_1(r_2r_3+r_1(r_2+r_3))))c^2+\big((p_3+p_1r_1r_2+p_1(r_1+r_2)r_3)b^2+\\ & (3p_4-3p_1r_1r_2r_3+p_3(r_1+r_2+r_3)-p_2(r_2r_3+r_1(r_2+r_3)))b+\\ & 2(p_2r_1r_2r_3+p_4(r_1+r_2+r_3)+p_3(r_2r_3+r_1(r_2+r_3))))\,c-\\ & bp_4(b+r_1)(b+r_2)+\big(\big(p_1r_1b^3-p_2r_1b^2-p_4b+p_3r_1b-2p_4r_1\big)\,r_2-\\ & bp_4(b+r_1)\big)\,r_3\big),\end{aligned}$$

and $\gamma_i=A_i/B_i=-(f(r_i)+a_0B_i)/[(c+br_i+r_i^2)\prod_{j=1,\,j\neq i}^3(r_i-r_j)]$ for $i=1,2,3$, $\gamma_4=A_4/B_4=A_4/[2\prod_{i=1}^3(c+b\,r_i+r_i^2)]$

$$\begin{aligned}A_4 = \;& -a_0B_4+5\big((p_2+p_1(r_1+r_2+r_3))c^2-(p_4+p_1r_1r_2r_3+\\ & p_3(r_1+r_2+r_3)+p_2(r_2r_3+r_1(r_2+r_3))+\\ & b(p_3-p_1(r_2r_3+r_1(r_2+r_3))))c+p_4(b+r_1)(b+r_2)+\\ & \big(p_4(b+r_1)+\big(p_4+\big(p_1b^2-p_2b+p_3\big)\,r_1\big)\,r_2\big)\,r_3\big).\end{aligned}$$

We note that an integer power of H is a polynomial if and only if $\beta=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2,3,4$ and they all have the same sign.

Case 9): A first integral H is

$$\begin{aligned}&(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x^2+bxy+cy^2)^{\gamma_3}\\ &\exp\left(-\frac{f(r_1)x}{r_1(r_1-r_2)(c+r_1\,b+r_1^2)(r_1y-x)}+\right.\\ &\left.\frac{\beta x}{\prod_{i=1}^2(c+b\,r_i+r_i^2)^{3-i}\sqrt{(4c-b^2)x^2}}\arctan\left(\frac{bx+2cy}{\sqrt{(4c-b^2)\,x^2}}\right)\right),\end{aligned}$$

where

$$\begin{aligned}
\beta = \;& 5\big(2p_1c^3 - (b(p_2 - p_1(2r_1 + r_2)) + 2(p_3 + p_2(2r_1 + r_2)+\\
& p_1r_1(r_1 + 2r_2)))c^2 + \big((p_3 + p_1r_1(r_1 + 2r_2))b^2+\\
& (3p_4 + p_3(2r_1 + r_2) - r_1(3p_1r_1r_2 + p_2(r_1 + 2r_2)))b+\\
& 2(p_4(2r_1 + r_2) + r_1(p_2r_1r_2 + p_3(r_1 + 2r_2))))\, c-\\
& bp_4(b + r_1)^2 + \left(-p_4b^2 - 2p_4r_1b + \left(b\left(p_1b^2 - p_2b + p_3\right) - 2p_4\right) r_1^2\right) r_2\big),\\
\gamma_1 = \;& -\frac{A_1}{B_1} = -\frac{A_1}{(r_1 - r_2)^2(c + b\, r_1 + r_1^2)^2},\\
A_1 = \;& 2a_0c^2(r_1 - r_2)^2 + 2a_0b^2r_1^2(r_1 - r_2)^2+\\
& c\big(- 5p_4 + r_1\left(4a_0(b + r_1)(r_1 - r_2)^2 + 5p_1r_1(2r_1 - 3r_2)+\right.\\
& \left.5p_2(r_1 - 2r_2)\right) - 5p_3r_2\big) + b\big(\left(4a_0r_1(r_1 - r_2)^2 - 5p_3+\right.\\
& \left.5p_1r_1(r_1 - 2r_2) - 5p_2r_2\right) r_1^2 + 5p_4(r_2 - 2r_1)\big) + r_1\big(5p_4(2r_2 - 3r_1)+\\
& r_1 \left(\left(2a_0(r_1 - r_2)^2 - 5p_2 - 5p_1r_2\right) r_1^2 + 5p_3(r_2 - 2r_1)\right) \big),\\
\gamma_2 = \;& \frac{A_2}{B_2} = \frac{-f(r_2) + a_0B_2}{(r_1 - r_2)^2(c + b\, r_2 + r_2^2)},\\
\gamma_3 = \;& \frac{A_3}{B_3} = \frac{A_3}{2(c + b\, r_1 + r_1^2)^2(c + b\, r_2 + r_2^2)},\\
A_3 = \;& -a_0B_3 + 5\big((p_2 + p_1(2r_1 + r_2))c^2 - (p_4 + p_3(2r_1 + r_2)+\\
& r_1(p_1r_1r_2 + p_2(r_1 + 2r_2)) + b(p_3 - p_1r_1(r_1 + 2r_2)))c+\\
& p_4(b + r_1)^2 + \left(\left(p_1b^2 - p_2b + p_3\right) r_1^2 + 2p_4r_1 + bp_4\right) r_2\big).
\end{aligned}$$

We note that an integer power of H is a polynomial if and only if $f(r_1) = 0$, $\beta = 0$ and $\gamma_i \in \mathbb{Q}$ for $i = 1, 2, 3$ and they all have the same sign.

Case 10): A first integral H is

$$(x - r_1y)^{\gamma_1}(x^2 + bxy + cy^2)^{\gamma_2} \exp\left(\frac{f(r_1)(c + b\, r_1 + r_1^2)^2x^2}{r_1^2(x - r_1y)^2}\right.$$
$$\left.-\frac{\beta_1(c + b\, r_1 + r_1^2)x}{r_1^2(x - r_1y)} + \frac{\beta_2x}{\sqrt{(4c - b^2)x^2}} \arctan\left(\frac{bx + 2cy}{\sqrt{(4c - b^2)\, x^2}}\right)\right),$$

where

$$\begin{aligned}
\beta_1 = \;& 10\big((p_2 - bp_1)r_1^4 + 2(p_3 - cp_1)r_1^3 + (-cp_2 + bp_3 + 3p_4)r_1^2+\\
& 2bp_4r_1 + cp_4\big),\\
\beta_2 = \;& 10\big(\left(p_1r_1^3 - p_4\right) b^3 - r_1 \left(p_2r_1^2 + 3p_4\right) b^2 + r_1^2(p_3r_1 - 3p_4)b-\\
& 2p_4r_1^3 + 2c^3p_1 - c^2(2p_3 + b(p_2 - 3p_1r_1) + 6r_1(p_2 + p_1r_1))+\\
& c\left(\left(3p_1r_1^2 + p_3\right) b^2 + 3(p_4 + r_1(p_3 - r_1(p_2 + p_1r_1)))b+\right.\\
& \left.2r_1(3p_4 + r_1(3p_3 + p_2r_1))\right) \big),
\end{aligned}$$

and

$$\begin{aligned}
\gamma_1 = \;& -2\big(3a_0(c + r_1(b + r_1))^3 + 5\left(\left(p_1b^2 - p_2b + p_3\right) r_1^3 + 3p_4r_1^2+\right.\\
& 3bp_4r_1 + b^2p_4 + c^2(p_2 + 3p_1r_1) - c\left(p_4 + b\left(p_3 - 3p_1r_1^2\right) +\right.\\
& \left.\left.r_1(3p_3 + r_1(3p_2 + p_1r_1))\right)\right) \big),\\
\gamma_2 = \;& 5\big(\left(p_1b^2 - p_2b + p_3\right) r_1^3 + 3p_4r_1^2 + 3bp_4r_1 + b^2p_4 + c^2(p_2 + 3p_1r_1)\\
& -c\left(p_4 + b\left(p_3 - 3p_1r_1^2\right) + r_1(3p_3 + r_1(3p_2 + p_1r_1))\right) \big)\\
& -2a_0(c + r_1(b + r_1))^3
\end{aligned}$$

We note that an integer power H is a polynomial if and only if $f(r_1)=0$, $\beta_1=\beta_2=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2$ and they all have the same sign.

Case 11): A first integral H is

$$(x^2+b_1xy+c_1y^2)^{\gamma_1}(x^2+b_2xy+c_2y^2)^{\gamma_2}(x-r_1y)^{\gamma_3}$$
$$\exp\left(\sum_{i=1}^{2}\frac{\beta_i x}{\sqrt{(4c_i-b_i^2)x^2}}\arctan\left(\frac{b_ix+2c_iy}{\sqrt{(4c_i-b_i^2)\,x^2}}\right)\right),$$

with $\beta_i=\alpha_i/\delta_i$ for $i=1,2$, where

$$\begin{aligned}\alpha_i = {} & 5(-b_i(c_i^2p_2+c_ic_jp_2+b_jc_i(c_ip_1+p_3)-3c_ip_4+c_jp_4)+\\ & b_i(c_i^2p_1+c_jp_3+c_i(-3c_jp_1+b_jp_2+p_3)+b_jp_4)r_1+\\ & b_i^3(-p_4+c_jp_1r_1)+b_i^2(c_ic_jp_1+c_ip_3+b_jp_4-(b_jc_ip_1+c_jp_2+p_4)r_1)+\\ & 2(c_i^3p_1-c_jp_4r_1-c_i^2(c_jp_1+p_3+p_2r_1-b_j(p_2+p_1r_1))+\\ & c_i(p_4r_1+c_j(p_3+p_2r_1)-b_j(p_4+p_3r_1))))),\\ \delta_i = {} & ((b_2^2c_1+(c_1-c_2)^2+b_1^2c_2-b_1b_2(c_1+c_2))(c_i+b_ir_1+r_1^2),\end{aligned}$$

for $i,j=1,2$ and $i\neq j$

$$\begin{aligned}\gamma_i = {} & -2a_0(b_j^2c_i+(c_i-c_j)^2+b_i^2c_j-b_ib_j(c_i+c_j))(c_i+r1(b_i+r1))+\\ & 5(b_i^2(p_4+c_jp_1r_1)+b_i(c_ic_jp_1-c_ip_3-c_jp_2r_1+p_4r_1)+\\ & (c_i-c_j)(-p_4-p_3r_1+c_i(p_2+p_1r_1))-b_j(c_i^2p_1+p_4(b_i+r_1)-\\ & c_i(p_3+(-b_ip_1+p_2)r_1))),\end{aligned}$$

for $i,j=1,2$ and $i\neq j$. Finally

$$\gamma_3=\frac{A_3}{B_3}=\frac{-(f(r_1)+a_0B_3)}{\prod_{i=1}^{2}(c_i+b_i\,r_1+r_1^2)}.$$

We note that an integer power of H is a polynomial if and only if $\beta_1=\beta_2=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2,3$ and they all have the same sign.

Case 12): A first integral H is

$$(x-ry)^{\gamma_1}(x^2+bxy+cy^2)^{\gamma_2}$$
$$\exp\left(\frac{\beta_1x^3}{[(4c-b^2)x^2]^{3/2}}\arctan\left(\frac{bx+2cy}{\sqrt{(4c-b^2)\,x^2}}\right)+\frac{10\beta_2(c+b\,r+r^2)x}{(b^2-4c)c(x^2+bxy+cy^2)}\right),$$

where

$$\begin{aligned}\beta_1 = {} & -10\big((p_4+r(p_3-r(p_2+p_1r)))b^3+4p_3r^2b^2+2r^2(p_3r-3p_4)b-\\ & 4p_4r^3+4c^3p_1+c^2(4(p_3+r(p_2+3p_1r))-2b(p_2-3p_1r))-\\ & 2c\left(2p_2rb^2+(3p_4-r(p_3+r(3p_1r-p_2)))b+2r(3p_4+r(p_3+p_2r))\right)\big),\\ \beta_2 = {} & -p_4yb^3+(c(p_1rx+p_3y)-p_4(x+ry))b^2+\big((p_1(x+ry)-p_2y)c^2+\\ & (-p_2rx+3p_4y+p_3(x+ry))c-p_4rx\big)b+\\ & 2c\left(p_1yc^2-(p_1rx+p_3y+p_2(x+ry))c+p_3rx+p_4(x+ry)\right),\\ \gamma_1 = {} & 2(a_0(c+br+r^2)^2+f(r)),\\ \gamma_2 = {} & 4a_0(c+br+r^2)^2-f(r).\end{aligned}$$

We note that an integer power of H is a polynomial if and only if $\beta_1=\beta_2=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2$ and they all have the same sign.

4. Weight exponent $\mathbf{s} = (1, 2)$

We prove Theorem 1. Since systems in (5) are homogeneous, we know that they are integrable because they have the inverse integrating factor $V = x\dot{y} - y\dot{x}$. The strategy will be to obtain such first integrals and determine whose of them are polynomials. Denoting systems in (5) by $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ the first integral is $H(x, y) = \int (P(x, y)/V(x, y))\, dy + g(x)$, satisfying $\partial H/\partial x = -Q(x, y)/V(x, y)$.

The first system in (5) has the first integral

$$H = x^{-3-2p_2}(3y^2 - x^2)^{-3+p_2} \exp\Big(- 2\sqrt{3}p_1 \operatorname{arctanh}\big(x/(\sqrt{3}y)\big)\Big).$$

Note that an integer power of H is a polynomial if and only if $p_1 = 0$ and $p_2 = 3(1 - q)/(1 + 2q)$ with $q \in \mathbb{Q}^+$.

The second system in (5) has the first integral

$$H = x^{-3-2p_2}(3y^2 + x^2)^{-3+p_2} \exp\Big(- 2\sqrt{3}p_1 \arctan\big(x/(\sqrt{3}y)\big)\Big).$$

Note that an integer power of H is a polynomial if and only if $p_1 = 0$ and $p_1 = 3(1 - q)/(1 + 2q)$ with $q \in \mathbb{Q}^+$.

The third system in (5) has the first integral

$$H = x^{-2(3+p_1)} y^{-3+2p_1} \exp(2p_2 y/x).$$

Note that an integer power of H is a polynomial if and only if $p_2 = 0$ and $p_2 = 3(1 - 2q)/(2(1 + q))$ with $q \in \mathbb{Q}^+$.

The fourth system in (5) has the first integral

$$H = x \exp(-y(2p_1 x + p_2 y)/(3x^2)).$$

Note that an integer power of H is a polynomial if and only if $p_1 = p_2 = 0$.

The fifth system in (5) has the first integral x/y which is never a polynomial.

5. Weight exponent $\mathbf{s} = (1, 3)$

Doing the change of variables $(X, Y) = (x^3, y)$ the planar weight homogeneous systems of weight degree 4 and weight exponent $(1, 3)$ becomes

$$\dot{X} = 3a_{40}X^2 + 3a_{11}XY, \quad \dot{Y} = b_{60}X^2 + b_{31}XY + b_{02}Y^2. \qquad (13)$$

Again we shall use the inverse integrating factor $V = X\dot{Y} - Y\dot{X}$ for computing the first integrals of system (13).

It is clear that if $a_{11} = a_{40} = 0$ then a polynomial first integral is X, and if $b_{60} = b_{31} = b_{02} = 0$ then a polynomial first integral is Y. Now we consider the other cases.

Case 1: $3a_{11} - b_{02} \neq 0$ and $R = -(3a_{40} - b_{31})^2 + 4(-3a_{11} + b_{02})b_{60} \neq 0$. In this case system (13) has the first integral

$$\frac{6}{\sqrt{R}}(a_{11}(3a_{40} + b_{31}) - 2a_{40}b_{02}) \arctan\left(\frac{3a_{40}X - b_{31}X + 6a_{11}Y - 2b_{02}Y}{\sqrt{R}X}\right) +$$
$$2(3a_{11} - b_{02}) \log X + 3a_{11} \log\left(\frac{Y(3a_{40}X - b_{31}X + 3a_{11}Y - b_{02}Y)}{X^2} - b_{60}\right).$$

Here $\log A$ always means $\log|A|$, and as usual log is the logarithm in base e. Since this first integral must be a polynomial we must have

$$a_{11}(3a_{40}+b_{31})-2a_{40}b_{02}=0. \tag{14}$$

We consider different subcases.

If $3a_{11}-2b_{02}\neq 0$. Then from (14) we get

$$a_{40}=-\frac{a_{11}b_{31}}{3a_{11}-2b_{02}}.$$

Therefore, doing the exponential of the previous first integral we obtain that the first integral is

$$H=X^{1-\frac{3a_{11}}{6a_{11}-2b_{02}}}p(X,Y)^{\frac{3a_{11}}{6a_{11}-2b_{02}}},$$

where

$$p(X,Y)=(3a_{11}-2b_{02})b_{60}X^2+2(3a_{11}-b_{02})b_{31}XY-(9a_{11}^2-9a_{11}b_{02}+2b_{02}^2)Y^2.$$

Note that since $a_{11}-b_{02}\neq 0$ and $3a_{11}-2b_{02}\neq 0$ we have $9a_{11}^2-9a_{11}b_{02}+2b_{02}^2\neq 0$. Therefore an integer power of H is a polynomial first integral if and only if $3a_{11}/(6a-11-2b_{02})=m/n\in\mathbb{Q}^+$, and $m/n<1$. In this case the first integral H is

$$X^{n-m}((3a_{11}-2b_{02})b_{60}X^2+2(3a_{11}-b_{02})b_{31}XY-(9a_{11}^2-9a_{11}b_{02}+2b_{02}^2)Y^2)^m.$$

If $3a_{11}-2b_{02}=0$, that is, $b_{02}=3a_{11}/2$. In this case from (14) we get $a_{11}b_{31}=0$. Hence either $a_{11}=0$, or $b_{31}=0$. But if $a_{11}=0$, then $b_{02}=0$ in contradiction with the fact that $3a_{11}-2b_{02}\neq 0$. Therefore, this case is not possible and we must have $b_{31}=0$. The first integral is then

$$H=\frac{-2b_{60}X^2+6a_{40}XY+3a_{11}Y^2}{X},$$

which is never a polynomial.

Case 2*:* $b_{02}=3a_{11}$ and $b_{31}-3a_{40}\neq 0$. In this case system (13) has the first integral

$$(b_{31}-3a_{40})^2\log X+\frac{3a_{11}(3a_{40}-b_{31})Y}{X}+$$
$$3(-3a_{40}^2+b_{31}a_{40}-a_{11}b_{60})\log\Big(-\frac{b_{60}X-3a_{40}Y+b_{31}Y}{X}\Big).$$

In order that the first integral is a polynomial we must have $a_{11}(3a_{40}-b_{31})=0$, that is $a_{11}=0$ (and hence $b_{02}=0$). Then doing the exponential of the previous first integral we obtain the first integral

$$X^{1/3}\Big(\frac{b_{60}X-3a_{40}Y+b_{31}Y}{X}\Big)^{\frac{a_{40}}{3a_{40}-b_{31}}}.$$

Then we must have $b_{31}/a_{40}=-m/n$ with $m/n\in\mathbb{Q}^+$. In this case the previous first integral becomes $H=X^m(b_{60}X-3a_{40}Y+b_{31}Y)^{3n}$.

Case 3*:* $b_{02}=3a_{11}$ and $b_{31}=3a_{40}$. Then system (13) has the first integral

$$\frac{-2b_{60}X^2\log X+6a_{40}YX+3a_{11}Y^2}{6X^2}.$$

Since the case $a_{40} = a_{11} = 0$ has been studied, we have that in this case the first integral is never a polynomial.

Case 4: $3a_{11} - b_{02} \neq 0$ and $R = 0$. Then

$$b_{06} = \frac{-(3a_{40} - b_{31})^2}{4(3a_{11} - b_{02})},$$

and system (13) has the first integral

$$\frac{3(-3a_{11}a_{40} + 2b_{02}a_{40} - a_{11}b_{31})X}{3a_{40}X - b_{31}X + 6a_{11}Y - 2b_{02}Y} + (3a_{11} - b_{02})\log(36a_{11}X - 12b_{02}X) +$$
$$3a_{11}\log\left(-\frac{3a_{40}X - b_{31}X + 6a_{11}Y - 2b_{02}Y}{X}\right).$$

In order that it is a polynomial we must have

$$2a_{40}b_{02} - a_{11}(3a_{40} + b_{31}) = 0. \tag{15}$$

We consider two different subcases.

If $3a_{11} \neq 2b_{02}$. In this case condition (15) becomes

$$a_{40} = -\frac{a_{11}b_{31}}{3a_{11} - 2b_{02}}.$$

Then doing the exponential of the previous first integral we obtain the first integral

$$X^{\frac{1}{36a_{11} - 12b_{02}}}\left(\frac{b_{31}X - 3a_{11}Y + 2b_{02}Y}{X}\right)^{\frac{a_{11}}{4(b_{02} - 3a_{11})^2}}.$$

From this first integral we obtain the first integral

$$X(b_{31}X - 3a_{11}Y + 2b_{02}Y)^{-3a_{11}/b_{02}}.$$

So, if $b_{02} \neq 0$ then $3a_{11}/b_{02} = -m/n$ with $m/n \in \mathbb{Q}^+$ and the polynomial first integral is

$$X^n(b_{31}X - 3a_{11}Y + 2b_{02}Y)^m.$$

If $b_{02} = 0$ then $H = b_{31}X - 3a_{11}Y$ is a polynomial first integral. This concludes the proof of the theorem.

6. Weight exponent $\mathbf{s} = (1, 4)$

We introduce the change $(X, Y) = (x^4, y)$ in the planar weight homogeneous polynomial differential systems (1) of weight degree 4 with weight exponent $(1, 4)$. In these new variables (X, Y) the systems with weight exponent $(1, 4)$ becomes, after introducing the new independent variable $d\tau = x^3\, dt$, as follows

$$X' = 4(a_{40}X + a_{01}Y), \quad Y' = b_{70}X + b_{31}Y, \tag{16}$$

where the prime denotes derivative with respect to τ. If $a_{40} = a_{01} = 0$ then a polynomial first integral is X, and if $b_{70} = b_{31} = 0$ then a polynomial first integral is Y. Now we consider the other cases.

Case 1: $R = -(4a_{40} - b_{31})^2 - 16a_{01}b_{70} \neq 0$. Then a first integral of system (16) is

$$\frac{2(4a_{40}+b_{31})}{\sqrt{R}} \arctan\left(\frac{(4a_{40}-b_{31})X + 8a_{01}Y}{\sqrt{R}X}\right) +$$
$$\log(-b_{70}X^2 + (4a_{40}-b_{31})XY + 4a_{01}Y^2).$$

Since it must be a polynomial we must have $b_{31} = -4a_{40}$, and we get the polynomial first integral is $H = -b_{70}X^2 + 8a_{40}XY + 4a_{01}Y^2$.

Case 2: $R = 0$. We consider different subcases.

We first study when $b_{70} \neq 0$. Then from $R = 0$ we get

$$a_{01} = -\frac{(4a_{40}-b_{31})^2}{16b_{70}}. \tag{17}$$

If $b_{31} - 4a_{40} \neq 0$ the first integral is

$$\frac{2(4a_{40}+b_{31})b_{70}X}{-2b_{70}X + (4a_{40}-b_{31})Y} + (4a_{40}-b_{31})\log(2b_{70}X + (-4a_{40}+b_{31})Y),$$

which is a polynomial if and only if $b_{31} = -4a_{40}$. The polynomial first integral is $b_{70}X - 4a_{40}Y$.

If $b_{31} = 4a_{40}$ then $a_{40} \neq 0$ (otherwise $b_{31} = 0$ and from (17) we also have $a_{01} = 0$ and this case has been considered), and the first integral of (16) is

$$H = \frac{Y}{X} - \frac{b_{70}\log X}{4a_{40}},$$

which is never a polynomial.

If $b_{70} = 0$ then from $R = 0$ we get $b_{31} = 4a_{40}$. We only consider the case $a_{40} \neq 0$, and the first integral of (16) is

$$-\frac{a_{40}X}{Y} + a_{01}\log Y,$$

which is never a polynomial. This completes the proof of the theorem.

7. Weight exponent $s = (2, 3)$

We prove Theorem 4. The planar weight homogeneous polynomial differential systems (1) with weight degree 4 with weight-exponent $(2, 3)$ are

$$\dot{x} = a_{11}xy, \quad \dot{y} = b_{30}x^3 + b_{02}y^2. \tag{18}$$

If $a_{11}(3a_{11} - 2b_{02}) \neq 0$ then the first integral of system (18) is

$$H = x^{-\frac{2b_{02}}{a_{11}}}(2b_{30}x^3 - 3a_{11}y^2 + 2b_{02}y^2).$$

Then an integer power of H is a polynomial first integral if and only if $-2b_{02}/a_{11} \in \mathbb{Q}^+$.

If $a_{11} = 0$ then $H = x$ is a polynomial first integral of system (18).

If $b_{30} = b_{02} = 0$ then $H = y$ is a polynomial first integral of system (18).

If $a_{11} \neq 0$ and $3a_{11} = 2b_{02}$, then the first integral of system (18) is

$$H = \frac{y^4}{x^3} - \frac{2b_{30}}{a_{11}}\log x,$$

which is never a polynomial. This completes the proof of the theorem.

8. Weight exponent $\mathbf{s} = (2, 5)$

We prove Theorem 5. The planar weight homogeneous polynomial differential systems (1) of weight degree 4 with weight exponent $(2, 5)$ are

$$\dot{x} = a_{01}y, \qquad \dot{y} = b_{40}x^4.$$

It is straightforward to prove that $H = 2b_{40}x^5 - 5a_{01}y^2$ is a polynomial first integral.

9. Weight exponent $\mathbf{s} = (6, 9)$

We prove Theorem 6. The planar weight homogeneous polynomial differential systems (1) of weight degree 4 with weight exponent $(6, 9)$ are

$$\dot{x} = a_{01}y, \qquad \dot{y} = b_{20}x^2.$$

It is straightforward to prove that $H = 2b_{20}x^3 - 3a_{01}y^2$ is a polynomial first integral.

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