DARBOUX INTEGRABILITY OF POLYNOMIAL DIFFERENTIAL SYSTEMS IN \mathbb{R}^3

JAUME LLIBRE¹ AND CLÀUDIA VALLS²

ABSTRACT. In this article we study the Darboux integrability of the polynomial differential systems

 $\dot{x} = y - x^2$, $\dot{y} = z - x$, $\dot{z} = -d - ax - by - cz$.

This system comes from the study of a Hopf bifurcation in slow-fast systems with two slow variables and one fast variable. The tools used here for studying the Darboux integrability can be applied to arbitrary polynomial differential systems in \mathbb{R}^3 .

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Recently the Hopf bifurcations have been studied intensively in two dimensional differential systems with one slow and one fast variable, see for instance [1, 5, 6, 10, 8]. Less analysis has been done of the Hopf bifurcations in slow-fast systems in \mathbb{R}^3 with two slow variables and one first variable, see [7, 9, 13, 14]. Guuckenheimer in [9] reduces the study of this Hopf bifurcation to study the zero Hopf bifurcation of the differential system

(1)
$$\begin{aligned} \dot{x} &= y - x^2, \\ \dot{y} &= z - x, \\ \dot{z} &= -d - ax - by - cz, \end{aligned}$$

where $(x, y, z) \in \mathbb{R}^3$, $a, b, c, d \in \mathbb{R}$ and the dot denotes derivative with respect to the independent variable t.

The vector field associated to (1) is

$$\mathcal{X} = (y - x^2)\frac{\partial}{\partial x} + (z - x)\frac{\partial}{\partial y} - (d + ax + by + cz)\frac{\partial}{\partial z}.$$

Let U be an open subset in \mathbb{R}^3 such that $\mathbb{R}^3 \setminus U$ has zero Lebesgue measure. We say that a real function $H = H(x, y, z) \colon U \subset \mathbb{R}^3 \to \mathbb{R}$ non-constant in any open subset of U is a first integral if H(x(t), y(t), z(t)) is constant on all solutions (x(t), y(t), z(t))of \mathcal{X} contained in U, i.e. $\mathcal{X}H_{|U} = 0$. The existence of a first integral for a differential system in \mathbb{R}^3 allows to reduce its study in one dimension. This is the main reason to look for first integrals.



²⁰¹⁰ Mathematics Subject Classification. Primary 34A05, 34A34, 34C14.

Key words and phrases. Darboux integrability, invariants, Darboux polynomial.