ANALYTIC INTEGRABILITY OF A CLASS OF PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS

INTÉGRABILITÉ ANALYTIQUE D'UNE CLASSE DE SYSTÈMES DIFFÉRENTIELS POLYNÔMIAUX DANS LE PLAN

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Abstract. In this paper we find necessary and sufficient conditions in order that the differential systems of the form $\dot{x} = xf(y)$, $\dot{y} = g(y)$, with f and g polynomials, have a first integral which is analytic in the variable x and meromorphic in the variable y. We also characterize their analytic first integrals in both variables x and y.

These polynomial differential systems are important because after a convenient change of variables they contain all quasi–homogeneous polynomial differential systems in $\mathbb{R}^2$.

Résumé. Dans cet article, nous trouvons des conditions nécessaires et suffisantes pour que les systèmes différentiels de la forme $\dot{x} = xf(y)$, $\dot{y} = g(y)$, avec f et g polynômes, ont une première intégrale qui est analytique dans la variable x et méromorphe dans la variable y. Nous caractérisons aussi leur intégrales première analytique dans les deux variables x et y.

Ces systèmes différentiels polynômiaux sont importants parce que, après une changement convenable de variables ils contiennent tous les systèmes différentiels polynômiaux quasi–homogènes en $\mathbb{R}^2$.

1. Introduction and statement of the main results

Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{C}[y]$ the ring of all polynomials in the variable y with coefficients in $\mathbb{C}$. In this paper we consider the polynomial differential systems of the form

$$\dot{x} = xf(y), \quad \dot{y} = g(y), \tag{1}$$

where $f, g \in \mathbb{C}[y]$ and are coprime. The dot denotes the derivative with respect to the independent variable t real or complex. We denote by $\mathcal{X} = (xf(y), g(y))$ the polynomial vector field associated to system (1), and we say that the degree of the system is $n = \max\{\deg\, xf(y), \deg\, g(y)\}$. For the sake of simplicity, we assume for the rest of the paper that system (1) is not linear, that is $n > 1$.

We recall that given a planar polynomial differential system (1), we say that a function $H\colon \mathcal{U} \subset \mathbb{C}^2 \to \mathbb{C}$ with $\mathcal{U}$ an open set, is a *first integral* of system (1) if

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H is continuous, not locally constant and constant on each trajectory of the system contained in $\mathcal{U}$. We note that if H is of class at least C^1 in $\mathcal{U}$, then H is a first integral if it is not locally constant and

$$xf(y)\frac{\partial H}{\partial x} + g(y)\frac{\partial H}{\partial y} = 0$$

in $\mathcal{U}$. We call the *integrability problem* the problem of finding such a first integral and the functional class where it belongs. We say the system has an *analytic first integral* if there exists a first integral $H(x, y)$ which is an analytic function in the variables x and y. We say that the system has a *pseudo–meromorphic first integral* if there exists a first integral $H(x, y)$ which is an analytic function in the variable x and a meromorphic function in the variable y.

The aim of this paper is to characterize the existence of first integrals of system (1) that can be described by functions that are analytic or pseudo–meromorphic.

Let α_l for $l = 1, \ldots, k$ be the zeros of g. We say that g is *square–free* if $g(y) = \prod_{l=1}^{k}(y - \alpha_l)$ with $\alpha_l \neq \alpha_j$ for $l, j = 1, \ldots, k$ and $l \neq j$. When g is square-free we define $\gamma_l = f(\alpha_l)/g'(\alpha_l)$ for $l = 1, \ldots, k$. With this notation we introduce the main result of the paper.

Theorem 1. *System (1) has a pseudo–meromorphic first integral if and only if $g(y)$ is square–free. Moreover, if $\gamma_l > 0$ for all $l = 1, \ldots, k$ then the first integral is analytic, otherwise it is a pseudo–meromorphic function with poles on $y = \alpha_l$ if $\gamma_l > 0$.*

The proof of Theorem 1 is given in section 2. Furthermore, the specific form of the first integral is given in the proof of Theorem 1.

Example 2. *Consider the differential system*

$$\dot{x} = xy^3, \quad \dot{y} = y + 1.$$

This system has the analytic first integral

$$H(x, y) = e^{-(y+1)(2y^2-5y+11)/6}x(y + 1).$$

Note that $g(y) = y + 1$ is square–free, $\alpha_1 = -1$ and $\gamma_1 = -1 < 0$.

Example 3. *Consider the differential system*

$$\dot{x} = xy^3, \quad \dot{y} = y - 1.$$

This system has the pseudo–meromorphic first integral

$$H(x, y) = \frac{e^{(1-y)(2y^2+5y+11)/6}x}{y - 1}.$$

Note that $g(y) = y - 1$ is square–free, $\alpha_1 = 1$ and $\gamma_1 = 1 > 0$.

System (1) is of separate variables and appears in many situations. In Lemma 2.2 of [1] it is proved that there exists a blow-up change of variables that transforms any quasi–homogeneous polynomial differential system into a differential system (1). However we point out that not all the planar polynomial differential systems (1) come from quasi–homogenous polynomial differential systems. We recall that a polynomial differential system

$$\dot{x} = P(x, y) \quad \dot{y} = Q(x, y)$$

is quasi-homogeneous if there exists $s_1, s_2, d \in \mathbb{N}$ (here $\mathbb{N}$ denotes the set of positive integers) such that for arbitrary $\alpha \in \mathbb{C}$,

$$P(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^{s_1 - 1 + d} P(x, y), \quad Q(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^{s_2 - 1 + d} Q(x, y). \tag{2}$$

From Theorem 3.1*b*) of [1] and Proposition 1 of [3] it follows the next result.

Theorem 4. *The quasi–homogeneous polynomial differential system* (2) *has an analytic first integral if and only if $g(y)$ is square–free,* $\deg f < \deg g$ *and* $\gamma_i \in \mathbb{Q}^-$ *for* $i = 1, 2, \ldots, k$ *and* $1 + \gamma_1 + \gamma_2 + \cdots + \gamma_k \geq 0$.

Note that Theorem 1 extends the result of Theorem 4 only valid for the quasi–homogeneous polynomial differential systems. Recall that quasi–homogeneous polynomial differential systems can be written as a subclass of the polynomial differential systems (1) using the Lemma 2.2 of [1].

2. Proof of Theorem 1

Assume that system (1) has a pseudo–meromorphic first integral. Then it can be written as a power series in x in the form

$$H(x, y) = \sum_{k \geq 0} a_k(y) x^k,$$

where $a_k(y)$ is a meromorphic function in the variable y. Then, it must satisfy

$$x f(y) \frac{\partial H}{\partial x} + g(y) \frac{\partial H}{\partial y} = 0, \tag{3}$$

that is

$$0 = \sum_{k \geq 0} k f(y) a_k(y) x^k + \sum_{k \geq 0} g(y) a_k'(y) x^k = \sum_{k \geq 0} \big(k f(y) a_k(y) + g(y) a_k'(y)\big) x^k.$$

Hence,

$$a_0'(y) = 0 \quad \text{that is} \quad a_0(y) = \text{constant}$$

and for $k \geq 1$,

$$k f(y) a_k(y) + g(y) a_k'(y) = 0 \quad \text{that is} \quad \frac{a_k'(y)}{a_k(y)} = \frac{-k f(y)}{g(y)}. \tag{4}$$

If $\deg f \geq \deg g$ and we consider the division of $-k f(y)$ by $g(y)$ we can write

$$k f(y) = q(y) g(y) + r(y),$$

where $r(y)$ cannot be zero taking into account that f and g are coprime and $\deg \psi < \deg g$. Hence equation (4) takes the form

$$\frac{a_k'(y)}{a_k(y)} = -q(y) - \frac{r(y)}{g(y)}. \tag{5}$$

Integrating this equation we have

$$a_k(y) = Ce^{-Q(y)}e^{-\int \frac{r(v)}{g(v)}\,dv} \tag{6}$$

where C is a constant of integration and $Q'(y) = q(y)$. Therefore, since the first factor of (6) is an analytic function, we must study the second factor in (6).

Assume that g is not square free. Using an affine transformation of the form $z = y + \alpha$ with $\alpha \in \mathbb{C}$ if it is necessary, we can assume that z is a multiple of g, that is, $\tilde{g}(z) = z^m R(z)$, where $\tilde{g}(z) = g(z-\alpha)$ with $m > 1$ an integer and $R(0) \neq 0$. Since f and g are coprime we also have $\tilde{r}(0) \neq 0$, where $\tilde{r}(z) = r(z-\alpha)$. Now we develop $\tilde{r}(z)/\tilde{g}(z)$ in simple fractions of z, that is,

$$\frac{\tilde{r}(z)}{\tilde{g}(z)} = \frac{c_m}{z^m} + \frac{c_{m-1}}{z^{m-1}} + \cdots + \frac{c_1}{z} + \frac{\alpha(z)}{R(z)},$$

where $\alpha(z)$ is a polynomial with deg $\alpha(z) <$ deg $R(z)$, and $c_i \in \mathbb{C}$ for $i = 1, \ldots, m$. Note that $c_m \neq 0$. Therefore integrating this last expression we have

$$\exp\left(\int \frac{\tilde{r}(z)}{\tilde{g}(z)}\,dz\right) = \exp\left(\frac{c_m}{(1-m)z^{m-1}}\right) \cdot \exp\left(\int \left(\frac{c_{m-1}}{z^{m-1}} + \cdots + \frac{c_1}{z} + \frac{\alpha(z)}{R(z)}\right) dz\right).$$

Note that the first exponential factor cannot be simplified by any part of the second exponential factor. Moreover $c_m \neq 0$ and we get a contradiction with the fact that the left hand side must be a meromorphic function in the variable y while $\exp(c_m/((1-m)z^{m-1})$ has an essential singularity at $z = 0$, and this it is not meromorphic in z. Therefore, we conclude that $g(y)$ is square-free. Hence we write

$$\frac{r(z)}{g(z)} = \frac{\gamma_1}{z-\alpha_1} + \cdots + \frac{\gamma_k}{z-\alpha_k}.$$

Then,

$$\int \frac{r(z)}{g(z)}\,dz = \sum_{j=0}^{k} \int \frac{\gamma_j}{z-\alpha_j}\,dz = \sum_{j=0}^{k} \gamma_j \log(z-\alpha_j)$$

and, consequently,

$$e^{\int \frac{r(z)}{g(z)}\,dz} = \prod_{j=0}^{k} (z-\alpha_j)^{\gamma_j}.$$

Note that this expression is always a meromorphic function. If $\gamma_j > 0$ for all $j = 1, \ldots, k$ then it is an analytic function in the variable y, otherwise it is meromorphic with poles on the α_j such that $\gamma_j < 0$. Hence $a_k(y)$ is an analytic function in y if $\gamma_j < 0$ for $j = 1, \ldots, k$, and it is meromorphic with poles on the α_j with $\gamma_j > 0$.

Conversely, assume that g is square–free and that $f(y) = q(y)g(y) + r(y)$. We will show that

$$H(x,y) = xe^{-\int q(y)\,dy}(y-\alpha_1)^{-\gamma_1} \cdots (y-\alpha_k)^{-\gamma_k},$$

with $\gamma_i = r(\alpha_i)/g'(\alpha_i)$ for $i = 1, \ldots, k$ is a pseudo–meromorphic function, and it is analytic if all $\gamma_j < 0$ for $j = 1, \ldots, k$. Now we must show that indeed it is a first integral of system (1). We set $\phi(y) = (y-\alpha_1)^{\gamma_1}\cdots(y-\alpha_k)^{\gamma_k}$. Note that

$$
\begin{aligned}
0 &= xf(y)\frac{\partial H}{\partial x} + g(y)\frac{\partial H}{\partial y}\\
&= xf(y)e^{-\int q(y)\,dy}\phi(y) + xg(y)(-q(y)\phi(y) - \phi'(y))e^{-\int q(y)\,dy}\\
&= xe^{-\int q(y)\,dy}\big(f(y)\phi(y) - g(y)q(y)\phi(y) - g(y)\phi'(y)\big)\\
&= xe^{-\int q(y)\,dy}\big(r(y)\phi(y) - g(y)\phi'(y)\big).
\end{aligned}
$$

To see that this last expression is identically zero it is equivalent to see that

$$\frac{\phi'(y)}{\phi(y)} = \frac{r(y)}{g(y)}.$$

Recalling the expression of $\phi(y)$ we have

$$\frac{\phi'(y)}{\phi(y)} = \frac{\gamma_1}{y-\alpha_1} + \frac{\gamma_2}{y-\alpha_2} + \cdots + \frac{\gamma_k}{y-\alpha_k}.$$

Taking common denominator and recalling that $g(y) = c(y-\alpha_1)(y-\alpha_2)\cdots(y-\alpha_k)$ we obtain

$$\frac{\phi'(y)}{\phi(y)} = \frac{c}{g(y)}\sum_{i=1}^{k}\gamma_i\prod_{j=1, j\neq i}^{k}(y-\alpha_j).$$

Now substituting the values of $\gamma_i = r(\alpha_i)/g'(\alpha_i)$ and taking into account that

$$g'(\alpha_i) = c\prod_{j=1, j\neq i}^{k}(\alpha_i - \alpha_j),$$

we get

$$\frac{\phi'(y)}{\phi(y)} = \frac{1}{g(y)}\sum_{i=1}^{k}r(\alpha_i)\prod_{j=1, j\neq i}^{k}\frac{y-\alpha_j}{\alpha_i-\alpha_j} = \frac{r(y)}{g(y)}. \tag{7}$$

The last expression in the sum, recalling that $\deg r < \deg g$, is the expression of the Lagrange polynomial which interpolates $r(y)$ in the k points $(\alpha_i, r(\alpha_i))$, for $i = 1, 2, \ldots, k$, see for more details [2]. Therefore this polynomial is $r(y)$, and we conclude that the expression (7) is satisfied. This completes the proof of the theorem.

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