HOPF BIFURCATION OF A GENERALIZED MOON-RAND SYSTEM

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ABSTRACT. We study the Hopf bifurcation from the equilibrium point at the origin of a generalized Moon-Rand system. We prove that the Hopf bifurcation can produce 8 limit cycles. The main tool for proving these results is the averaging theory of fourth order. The computations are not difficult, but very big and have been done with the help of Mathematica and Mapple.

1. Introduction and statement of the main result

The Moon-Rand systems were developed to model the control of flexible space structures (see [8, 2, 6]). They were introduced by Moon and Rand. It is a differential equation in $\mathbb{R}^3$ of the form

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u - uw, \\ \dot{w} &= -\lambda w + \sum_{i+j=2} c_i u^i v^{2-i}, \end{aligned} \tag{1}$$

where c_i are real parameters. In [6] Mahdi, Romanovski and Shafer studied the Hopf bifurcation of the equilibrium point localized at the origin of system (1) using the reduction to the center manifold and studying on this surface the Hopf bifurcation. They found that 2 limit cycles can bifurcate from the origin of system (1)-

In this paper we study the Hopf bifurcation of the equilibrium point localized at the origin of the generalized Moon-Rand systems

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u - uw, \\ \dot{w} &= -\lambda w + \sum_{i+j=2} b_{ij} u^i v^j w^{2-i-j} + \sum_{i+j=3} c_{ij} u^i v^j w^{3-i-j}, \end{aligned} \tag{2}$$

where b_{ij} and c_{ij} are real parameters. Our study of this Hopf bifurcation uses a complete different approach than the one given by Mahdi, Romanovski and Shafer [6]. Namely, we use here the averaging theory of fourth order, and we find that 8 limit cycles can bifurcate from the origin of system (2).

In the qualitative theory of differential equations the study of the limit cycles is one of the main topics. We recall that in bifurcation theory, a *Poincaré-Andronov-Hopf bifurcation* or simply a *Hopf bifurcation* of a differential system is a local bifurcation in which the equilibrium point of the differential system loses stability when a pair of complex conjugate eigenvalues of the linearization of the system

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around the equilibrium point cross the imaginary axis of the complex plane. Under appropriate assumptions on the differential system, small amplitude limit cycles bifurcate from the equilibrium point. We recall that a *limit cycle* is a periodic orbit isolated in the set of all the periodic orbits of the system.

In order to study the Hopf bifurcation at the origin of system (2) we choose the parameters of this system as follows.

$$
\begin{aligned}
\dot u &= v,\\
\dot v &= -u - uw,\\
\dot w &= -w\sum_{j=0}^{n}\varepsilon^j\lambda_j + u^2\sum_{j=0}^{n}\varepsilon^j b_{20j} + uv\sum_{j=0}^{n}\varepsilon^j b_{11j} + uw\sum_{j=0}^{n}\varepsilon^j b_{10j}\\
&\quad + v^2\sum_{j=0}^{n}\varepsilon^j b_{02j} + vw\sum_{j=0}^{n}\varepsilon^j b_{01j} + w^2\sum_{j=0}^{n}\varepsilon^j b_{00j} + u^3\sum_{j=0}^{n}\varepsilon^j c_{30j}\\
&\quad + u^2v\sum_{j=0}^{n}\varepsilon^j c_{21j} + u^2w\sum_{j=0}^{n}\varepsilon^j c_{20j} + uv^2\sum_{j=0}^{n}\varepsilon^j c_{12j} + uvw\sum_{j=0}^{n}\varepsilon^j c_{11j}\\
&\quad + uw^2\sum_{j=0}^{n}\varepsilon^j c_{10j} + v^3\sum_{j=0}^{n}\varepsilon^j c_{03j} + v^2w\sum_{j=0}^{n}\varepsilon^j c_{02j} + vw^2\sum_{j=0}^{n}\varepsilon^j c_{01j}\\
&\quad + w^3\sum_{j=0}^{n}\varepsilon^j c_{00j},
\end{aligned}
\tag{3}
$$

where ε is a small parameter and $n \geq 4$. We recall that the linear part of this system at the equilibrium point located at the origin has eigenvalues $\pm i$ and $-\sum_{j=0}^{n}\varepsilon^j\lambda_j$. Our main result is the following.

Theorem 1. *System* (3) *using averaging theory of fourth order can have* 8 *limit cycles bifurcating from the origin.*

The proof of Theorem 1 is given in section 2. It uses Theorem 2 of the appendix.

The method here followed, based in the averaging theory of fourth order, can be applied to many other differential systems.

There are many results on the Hopf bifurcation, probably the more classical one can be found in the book of Marsden and McCracken [7], see also [1]. But for studying the Hopf bifurcation here we shall use the averaging theory of fourth order which is rarely used in this context. We summarize the results that we need from the averaging theory in Appendix 1.

2. Proof of Theorem 1

To study the Hopf bifurcation at the origin (i.e. to study the small amplitude limit cycles bifurcating from the equilibrium point at the origin) we introduce cylindrical coordinates (r, θ, z) defined through $u = r\cos\theta$, $v = r\sin\theta$ and $w = z$. Note that $r \in (0, \infty)$, $\theta \in \Sigma^1 = \mathbb{R}/(2\pi\mathbb{R})$ and $z \in \mathbb{R}$. In order to study the small amplitude limit cycles around the origin of coordinates we do the rescaling $(r, \theta, z) = (\varepsilon R, \theta, \varepsilon Z)$. Now the system is written as $(R', Z') = (dR/d\theta, dZ/d\theta)$ and, after taking $\lambda_0 = 0$

(otherwise the system is not written in the appropriate normal form of averaging, see equation (8)) we obtain system (3) in the normal form of averaging, i.e.

$$
\begin{aligned}
R' &= \varepsilon F_{11} + \varepsilon^2 F_{21} + \varepsilon^3 F_{31} + \varepsilon^4 F_{41} + O(\varepsilon^5),\\
Z' &= \varepsilon F_{12} + \varepsilon^2 F_{22} + \varepsilon^3 F_{32} + \varepsilon^4 F_{42} + O(\varepsilon^5),
\end{aligned} \tag{4}
$$

where $F_{ij} = F_{ij}(\theta, R, Z)$ for $i = 1, \dots, 4$ and $j = 1, 2$ are given in Appendix 3 (due to their length) and $O(\varepsilon^5)$ denotes terms of order greater than or equal to 5 in ε. Using the notation of Appendix 1, we have that $x = (R, Z)$, $t = \theta$, $T = 2\pi$, $F_j = (F_{j1}, F_{j2})$ for $j = 1, \dots, 4$ and $\varepsilon^5 R = O(\varepsilon^5)$.

We note that the periodic solutions $(R(\theta), Z(\theta))$ of system (4) that we shall obtain using the averaging theory of fourth order, going back through the change of variables tends to the origin of system (3) when ε tends to zero, because $(r(\theta), z(\theta)) = (\varepsilon R(\theta), \varepsilon Z(\theta))$.

Now we compute the function $f_1(x) = f_1(R, Z) = (f_{11}(R, Z), f_{12}(R, Z))$ and we obtain that $f_1 = f_1(R, Z)$ is equal to

$$
f_1 = \big(0, -\pi((b_{020} + b_{200})R^2 + 2b_{000}Z^2 - 2\lambda_1 Z)\big).
$$

Note that f_{11} is identically zero and thus the averaged function of first order provides no limit cycles. We need to make f_1 identically zero. For that, we take

$$
b_{200} = -b_{020}, \quad b_{000} = 0, \quad \lambda_1 = 0.
$$

With this choice of the parameters we can compute the averaged function of second order, for more details see Theorem 2 in Appendix 1.

The averaged function of second order $f_2(x) = f_2(R, Z) = (f_{21}(R, Z), f_{22}(R, Z))$ has the two components

$$
f_2 = \pi\left(\frac{1}{4}b_{020}R^3, 2\lambda_2 Z - (b_{021} + b_{201})R^2 - 2b_{001}Z^2 - (b_{020} + c_{020} + c_{200})R^2 Z - 2c_{000}Z^3\right).
$$

In order to look for small amplitude limit cycles bifurcating from the origin, after all the changes of coordinates that we did and according to Theorem 2, we must find the zeros (R_0, Z_0) of the system $f_2(R, Z) = 0$ such that the Jacobian

$$
\det \begin{pmatrix} D_R f_{21} & D_Z f_{21} \\ D_R f_{22} & D_Z f_{22} \end{pmatrix}\Bigg|_{(R,Z)=(R_0,Z_0)} \neq 0.
$$

Note that the unique solution of $f_{21} = 0$ is $R = R_0 = 0$ and for this value of R the Jacobian is zero. Therefore, the averaged function of second order does not provide any small amplitude limit cycle and consequently we must choose the parameters to make f_{22} identically zero. Doing that we get

$$
b_{020} = 0, \quad b_{201} = -b_{021}, \quad c_{020} = -c_{200}, \quad b_{001} = 0, \quad c_{000} = 0, \quad \lambda_2 = 0.
$$

Now we compute the averaged function of third order, for more details see again Appendix 1.

The averaged function of third order $f_3(x) = f_3(R, Z) = (f_{31}(R, Z), f_{32}(R, Z))$ has the two components

$$
\begin{aligned}
f_{31} &= \frac{\pi}{4}\big(b_{021}R^3 - (b_{010}b_{100} + c_{200})R^3 Z\big),\\
f_{32} &= -\frac{\pi}{4}\big(4(b_{022} + b_{202})R^2 + (-b_{010}b_{100}b_{110} + 3b_{100}c_{030} - b_{010}c_{120} + b_{110}c_{200}\\
&\quad + b_{100}c_{210} - 3b_{010}c_{300})R^4 + 4(b_{021} + c_{021} + c_{201})R^2 Z + 8b_{002}Z^2\\
&\quad - 4(b_{010}b_{100} + b_{100}c_{010} - b_{010}c_{100} + c_{200})R^2Z^2 + 8c_{001}Z^3 - 8\lambda_3 Z\big).
\end{aligned}
$$

According to Theorem 2 we must find the zeros (R_0, Z_0) of the system $f_3(R, Z) = 0$ such that $R_0 > 0$ satisfying that

$$
\det\begin{pmatrix} D_R f_{31} & D_Z f_{31}\\ D_R f_{32} & D_Z f_{32}\end{pmatrix}\Bigg|_{(R,Z)=(R_0,Z_0)} \neq 0. \tag{5}
$$

It is easy to see that $f_3 = 0$ has at most two solutions satisfying (5) by choosing conveniently the values of the parameters of system (3). More precisely, note that $f_{31} = 0$ has a unique solution $Z = Z_0 = b_{021}/(b_{010}b_{100} + c_{200})$. Substituting this value of Z in f_{32} we get a polynomial of second order in the variable R^2. Then setting $f_{32} = 0$ we get that at most two positive values of $R = R_0$ are obtained such that the Jacobian is nonzero. Moreover, playing with the coefficients it follows easily that the upper bound can be reached. Since we are obtaining an upper bound on the number of limit cycles equal to the one in [6] we will continue with the averaging process. Hence, we choose the values of the parameters such that $f_3 = 0$. We have

$$
b_{021} = 0,\ c_{200} = -b_{010}b_{100},\ b_{202} = -b_{022},\ c_{021} = -c_{201},\ b_{002} = 0,\ c_{001} = 0,\ \lambda_3 = 0,
$$

and three possible solutions:

$$
\begin{aligned}
S_1 &= \big\{b_{010} = 0, b_{100} = 0\big\},\\
S_2 &= \big\{c_{100} = 0, c_{120} = -3c_{300}, b_{100} = 0\big\},\\
S_3 &= \Big\{c_{010} = \frac{b_{010}c_{100}}{b_{100}}, c_{030} = \frac{2b_{010}b_{100}b_{110} + b_{010}c_{120} - b_{100}c_{210} + 3b_{010}c_{300}}{3b_{100}}\Big\}.
\end{aligned}
$$

We will study separately each one of the possible solutions.

Case 1*: Solution* S_1. In this case the averaged function of fourth order (see again Theorem 2 in Appendix 1) yields $f_4(x) = f_4(R, Z) = (f_{41}(R, Z), f_{42}(R, Z))$ of the form

$$
\begin{aligned}
f_{41} &= \frac{\pi}{4}R^3(b_{022} - c_{201}Z),\\
f_{42} &= -\frac{\pi}{4}\big(4(b_{023} + b_{203})R^2 + (3b_{101}c_{030} - b_{011}c_{120} + b_{110}c_{201} + b_{101}c_{210} - 3b_{011}c_{300})R^4\\
&\quad + 4(b_{022} + c_{022} + c_{202})R^2 Z + 2(3c_{030}c_{100} - c_{010}c_{120} + c_{100}c_{210} - 3c_{010}c_{300})R^4 Z\\
&\quad + 8b_{003}Z^2 - 4(b_{101}c_{010} - b_{011}c_{100} + c_{201})R^2Z^2 + 8c_{002}Z^3 - 8\lambda_4 Z\big).
\end{aligned}
$$

According to Theorem 2 we must find the zeros (R_0, Z_0) of the system $f_4 = 0$ such that the Jacobian

$$
\det\begin{pmatrix} D_R f_{41} & D_Z f_{41}\\ D_R f_{42} & D_Z f_{42}\end{pmatrix}\Bigg|_{(R,Z)=(R_0,Z_0)} \neq 0. \tag{6}
$$

Proceeding as we did for the averaged function of third order, it is easy to see that $f_4 = 0$ has at most two solutions satisfying (6). Moreover, playing with the coefficients it follows easily that the upper bound can be reached.

Case 2*: Solution* S_2. In this case the averaged solution of fourth order is

$$\begin{aligned}
f_{41} &= \frac{\pi}{24}R^3\big(6b_{022} + b_{010}c_{300}R^2 - 6(b_{010}b_{101} + c_{201})Z\big),\\
f_{42} &= -\frac{\pi}{4}\big(4(b_{023} + b_{203})R^2 + (3b_{101}c_{030} + b_{110}c_{201} + b_{101}c_{210} - b_{010}(b_{101}b_{110}\\
&\quad + c_{121} + 3c_{301}))R^4 + 4(b_{022} + c_{022} + c_{202})R^2Z + b_{010}c_{300}R^4Z + 8b_{003}Z^2\\
&\quad + (4b_{010}(-b_{101} + c_{101}) - 4(b_{101}c_{010} + c_{201}))R^2Z^2 + 8c_{002}Z^3 - 8\lambda_4 Z\big).
\end{aligned}$$

We must find the zeros (R_0, Z_0) of the system $f_4 = 0$ such that the Jacobian (6) is nonzero. We will see that $f_4 = 0$ has at most three solutions satisfying (6) and that this upper bound is reached choosing adequately the values of the parameters of system (2). Indeed, $f_{41} = 0$ has a unique solution $Z = Z_0(R) = (6b_{022} + b_{010}c_{300}R^2)/(6(b_{010}b_{101} + c_{201}))$. Then substituting this value of Z into f_{42} we get a polynomial $g = g(R^2)$ of degree three in the variable R^2. Thus, making $f_{42} = 0$ and since we are looking for $R_0 > 0$ we get that $f_{42} = 0$ has at most three positive solutions $R = R_{0,i}$, $i = 1, 2, 3$. To see that this upper bound is reached we must show that we can choose the parameters in our system (3) such that the coefficients of the polynomial g are independent, and this is the case looking at these coefficients. So the maximum of three limit cycles can be reached.

Case 3*: Solution* S_3. In this case the averaged solution of fourth order is given by

$$\begin{aligned}
f_{41} &= -\frac{\pi}{72}R^3\big((-18b_{022} + (b_{100}c_{210} + b_{010}(4b_{100}b_{110} + 5c_{120} + 12c_{300}))R^2\\
&\quad + 18(b_{011}b_{100} + b_{010}b_{101} + c_{201})Z + 18b_{010}(-2b_{100} + 3c_{100})Z^2)\big),\\
f_{42} &= \frac{\pi}{24b_{100}}\big(- 24b_{100}(b_{023} + b_{203})R^2 + 6(b_{011}b_{100}^2b_{110} - b_{010}b_{100}b_{101}b_{110}\\
&\quad + 2b_{010}b_{100}^2b_{111} - 3b_{100}^2c_{031} + b_{011}b_{100}c_{120} - b_{010}b_{101}c_{120} + b_{010}b_{100}c_{121}\\
&\quad - b_{100}b_{110}c_{201} - b_{100}^2c_{211} + 3b_{011}b_{100}c_{300} - 3b_{010}b_{101}c_{300} + 3b_{010}b_{100}c_{301})R^4\\
&\quad - 24b_{100}(b_{022} + c_{022} + c_{202})R^2Z + 2b_{100}(7b_{010}b_{100}b_{110} + 8b_{010}c_{120} + b_{100}c_{210}\\
&\quad + 21b_{010}c_{300})R^4Z - 48b_{003}b_{100}Z^2 + 24(b_{011}b_{100}^2 + b_{010}b_{100}b_{101} + b_{100}^2c_{011}\\
&\quad - b_{011}b_{100}c_{100} + b_{010}b_{101}c_{100} - b_{010}b_{100}c_{101} + b_{100}c_{201})R^2Z^2 + 24b_{010}^2b_{100}^2R^3Z^2\\
&\quad - 48b_{100}c_{002}Z^3 - 3b_{010}b_{100}(7b_{100} - 12c_{100})R^2Z^3 + 48b_{100}\lambda_4 Z\big).
\end{aligned}$$

According to Theorem 2 we must find the zeros (R_0, Z_0) of the system $f_4 = 0$ such that the Jacobian in (6) be nonzero. We will see that $f_4 = 0$ can have 8 limit cycles by choosing in a convenient way the values of the parameters of system (3). Indeed, solving $f_{41} = 0$ in $R = R_0 = R_0(Z)$ we get

$$R_0 = \pm 3\sqrt{2}\sqrt{\frac{(2b_{010}b_{100} - 3b_{010}c_{100})Z^2 - (b_{011}b_{100} + b_{010}b_{101} + c_{201})Z + b_{022}}{4b_{010}b_{100}b_{110} + 5b_{010}c_{120} + b_{100}c_{210} + 12b_{010}c_{300}}}.$$

Since $R_0>0$ we will restrict to the above positive R_0. Substituting this value of R_0 into f_{42} we get the function

$$f_{42}(Z)=\sum_{i=0}^{8}\alpha_i f_i(Z) \tag{7}$$

where

$$\begin{aligned}
f_0(Z)=&\ \ 1,\\
f_1(Z)=&\ \ Z,\\
f_2(Z)=&\ \ Z^2,\\
f_3(Z)=&\ \ Z^3,\\
f_4(Z)=&\ \ Z^4,\\
f_5(Z)=&\ \ Z^5,\\
f_6(Z)=&\ \ Z^2\sqrt{(2b_{010}b_{100}-3b_{010}c_{100})Z^2-(b_{011}b_{100}+b_{010}b_{101}+c_{201})Z+b_{022}},\\
f_7(Z)=&\ \ Z^3\sqrt{(2b_{010}b_{100}-3b_{010}c_{100})Z^2-(b_{011}b_{100}+b_{010}b_{101}+c_{201})Z+b_{022}},\\
f_8(Z)=&\ \ Z^4\sqrt{(2b_{010}b_{100}-3b_{010}c_{100})Z^2-(b_{011}b_{100}+b_{010}b_{101}+c_{201})Z+b_{022}},
\end{aligned}$$

and the α_i for $i=0,1,\ldots,8$ are functions in the coefficients b_{003}, b_{010}, b_{011}, b_{022}, b_{023}, b_{100}, b_{101}, b_{110}, b_{111}, b_{203}, c_{002}, c_{011}, c_{022}, c_{031}, c_{100}, c_{101}, c_{120}, c_{121}, c_{201}, c_{202}, c_{210}, c_{211}, c_{300}, c_{301} and λ_4. We do not provide the explicit expressions of these functions because we shall need several pages for showing them. Their expressions are easy to compute with the help of an algebraic manipulator as Mathematica or Mapple.

We claim that the functions α_i for $i=0,1,\ldots,8$ are linearly independent. Now we shall prove the claim. The coefficient λ_4, b_{003} and c_{003} only appears in the functions α_1, α_2 and α_3 respectively. So these functions are independent of the other six functions. Once we have removed the functions α_1, α_2 and α_3 from the set of nine functions, we see that the coefficient b_{023} only appears in the function α_0, hence the function α_0 is independent of the five remaining functions. We remove the function α_0 from the set of the six functions, and it remains only the functions α_i for $i=4,5,6,7,8$ for showing that they are independent. Now the coefficient c_{301} only appears in the function α_4. Therefore we reduce the set to the functions α_i for $i=5,6,7,8$. The coefficient b_{022} only appears in the function α_6, and we reduce the set to the functions α_i for $i=5,7,8$. The coefficient b_{011} only appears in the function α_7, and we reduce the set to the functions α_i for $i=5,8$. Finally, the gradients of the functions α_5 and α_8 which only depend on the coefficients b_{010}, b_{100}, c_{100}, b_{110}, c_{120}, c_{210} and c_{300}, form a 2×7 matrix of rank 2, so they are independent. Hence the claim is proved.

The quadratic polynomial

$$(2b_{010}b_{100}-3b_{010}c_{100})Z^2-(b_{011}b_{100}+b_{010}b_{101}+c_{201})Z+b_{022}$$

in Z is always positive if $b_{022}>0$ and

$$4b_{010}b_{022}(3c_{100}-2b_{100})+(b_{011}b_{100}+b_{010}b_{101}+c_{201})^2>0.$$

Since we always can choose the coefficients b_{ijk} and c_{ijk} satisfying these two previous conditions, the square root

$$\sqrt{(2b_{010}b_{100} - 3b_{010}c_{100})Z^2 - (b_{011}b_{100} + b_{010}b_{101} + c_{201})Z + b_{022}},$$

will be always real for all values of $Z \in \mathbb{R}$. Moreover, note that additionally we can choose the coefficients b_{ijk} and c_{ijk} in order that the expression

$$4b_{010}b_{100}b_{110} + 5b_{010}c_{120} + b_{100}c_{210} + 12b_{010}c_{300}$$

be positive, so the function $R_0(Z)$ is real for all $Z \in \mathbb{R}$. In what follows we shall work only with coefficients b_{ijk} and c_{ijk} for which the mentioned square root and the function $R_0(Z)$ are real. In particular, the functions f_6, f_7 and f_8 are well defined in $\mathbb{R}$.

It is easy to check that the functions f_i for $i =, 1, \ldots, 8$ are linearly independent in $\mathbb{R}$, and since the coefficients α_i for $i = 0, 1, \ldots, 8$ are also independent, the function $f_{42}(Z)$ given in (7) satisfies the assumptions of Proposition 3 so there exists $r_i \in Z$ for $i = 1, \ldots, 8$ and α_j for $j = 0, 1, \ldots, 8$ such that $f_{42}(r_i) = 0$. Therefore we have eight zeros of the function $f_{42}(Z)$. Looking at the proof of Proposition 3 in [4] it follows that these solutions r_i are simple solutions of $f_{42}(Z)$. For each $Z_0 = r_i$ we have that $R_0(r_i)$ is a positive solution of $f_{41}(Z)$. So $(R, Z) = (R_0(r_i)), r_i)$ is a solution of $f_4(R, Z) = 0$. It is easy to check that the Jacobian (6) is not zero. Hence by Theorem 2 the differential system (4) has eight periodic solutions.

Appendix 1: Averaging theory of fourth order

We briefly recall the basic elements of the averaging theory of fourth order to establish the existence of periodic orbits. Roughly speaking, the method gives a quantitative relation between the solutions of a nonautonomous periodic system and the solutions of its averaged system, which is autonomous. The following theorem provides a fourth-order approximation for periodic solutions of the original system.

We consider the initial value problem

$$\dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 F_3(t, x) + \varepsilon^4 F_4(t, x) + \varepsilon^5 R(t, x, \varepsilon), \tag{8}$$

where $F_j \colon \mathbb{R} \times \Omega \to \mathbb{R}^n$ are such that for each $t \in \mathbb{R}$, $F_j(t, \cdot) \in C^{4-j}$ for $j = 1, \ldots, 4$ and $\partial^{4-j} F_j$ is locally Lipschitz in the second variable for $j = 1, \ldots, 4$. Moreover $R \colon \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is continuous and locally Lipschitz in the second variable. We assume that F_1, F_2, F_3, F_4, R are periodic of period T in the variable t and we set

$$\begin{aligned}
y_1(t, x) &= \int_0^t F_1(s, x)\, ds,\\
y_2(t, x) &= 2 \int_0^t \big(F_2(s, x) + (D_x F_1(s, x)) y_1(s, x)\big)\, ds,\\
y_3(t, x) &= 3 \int_0^t \big(2F_3(s, x) + 2(D_x F_2(s, x)) y_1(s, x) + (D_x F_1(s, x)) y_2(s, x)\\
&\qquad + (D^2_{xx} F_1(s, x))(y_1(s, x), y_1(s, x))\big)\, ds
\end{aligned}$$

and

$$
\begin{aligned}
f_1(x) &= \int_0^T F_1(t,x)\,dt,\\
f_2(x) &= \int_0^T \big(F_2(t,x)\,dt + (D_xF_1(t,x))y_1(t,x)\big)\,dt,\\
f_3(x) &= \int_0^T \Big(F_3(t,x) + (D_xF_2(t,x))y_1(t,x) + \frac{1}{2}(D^2_{xx}F_1(t,x))(y_1(t,x),y_1(t,x))\\
&\quad + \frac{1}{2}(D_xF_1(t,x))y_2(t,x)\Big)\,dt,\\
f_4(x) &= \int_0^T \Big(F_4(t,x) + (D_xF_3(t,x))y_1(t,x) + \frac{1}{2}(D_xF_2(t,x))y_2(t,x)\\
&\quad + \frac{1}{2}(D^2_{xx}F_2(t,x))(y_1(t,x),y_1(t,x)) + \frac{1}{6}(D_xF_1(t,x))y_3(t,x)\\
&\quad + \frac{1}{2}(D^2_{xx}F_1(t,x))(y_1(t,x),y_2(t,x))\\
&\quad + \frac{1}{6}(D^3_{xxx}F_1(t,x))(y_1(t,x),y_1(t,x),y_1(t,x))\Big)\,dt.
\end{aligned}
$$

We denote by $d_B(h,V,a)$ the Brouwer degree of h at some neighborhood V of a (see [5] for the definition). In order to see that $d_B(h,V,a)\neq 0$ it is sufficient to check that the Jacobian of $D_zh(z)$ at $z=a$ is not zero, see again [5] for more details.

Theorem 2. *Assume that $f_i(x)\equiv 0$ for $i=1,2,3$ and that $f_4(x)\not\equiv 0$. If $f_4(a)=0$ for some $a\in\Omega$ and there exists a neighborhood $V\subset\Omega$ of a such that $f_4(x)\neq 0$ for all $x\in\overline{V}\setminus\{a\}$, and that $d_B(f_4(x),V,a)\neq 0$, then for $|\varepsilon|>0$ sufficiently small, there exist a T-periodic solution $x(t,\varepsilon)$ of (8) such that $x(0,\varepsilon)\to a$ when $\varepsilon\to 0$.*

For a proof of Theorem 2 see Theorem A of [3].

Appendix 2: Auxiliary result

Now we introduce an auxiliary result (for a proof see, for instance, Proposition 1 of [4]). To state it we recall that given A a set and $f_0,f_1,\dots,f_n\colon A\to\mathbb{R}$, we say that $f_0,f_1,\dots,f_n$ are *linearly independent functions* if and only if for all $a\in A$, it holds

$$
\sum_{i=0}^{n}\alpha_i f_i(r)=0 \quad \Rightarrow \alpha_0=\alpha_1=\cdots=\alpha_n=0.
$$

Proposition 3. *If $f_0,f_1,\dots,f_n\colon A\to\mathbb{R}$ are linearly independent then there exist $r_1,\dots,r_n\in A$ and $\alpha_0,\alpha_1,\dots,\alpha_n\in\mathbb{R}$ such that for every $i\in\{1,\dots,n\}$,*

$$
\sum_{k=0}^{n}\alpha_k f_k(r_i)=0.
$$

APPENDIX 3: THE FUNCTIONS F_{ij}

Here we provide the functions $F_{ij} = F_{ij}(\theta, R, Z)$ which in the system (4):

$$
\begin{aligned}
F_{11} = \;& RZ\sin\theta\cos\theta,\\
F_{12} = \;& \frac{R^2}{2}(\cos(2\theta)b_{020} - b_{020} - b_{200} - b_{200}\cos(2\theta) - b_{110}\sin(2\theta))\\
& +Z(-b_{100}\cos\theta - b_{010}\sin\theta)R - b_{000}z^2 + \lambda_1 Z,
\end{aligned}
$$

$$
\begin{aligned}
F_{21} = \;& -RZ^2\sin\theta\cos^3\theta,\\
F_{22} = \;& \frac{R^3}{4}(-c_{120}\cos\theta - 3c_{300}\cos\theta + c_{120}\cos(3\theta) - c_{300}\cos(3\theta) - 3c_{030}\sin\theta\\
& -c_{210}\sin\theta + c_{030}\sin(3\theta) - c_{210}\sin(3\theta)) + R^2\Big(\frac{1}{2}(b_{021}\cos(2\theta) - b_{021}\\
& -b_{201} - b_{201}\cos(2\theta) - b_{111}\sin(2\theta)) + \frac{Z}{8}(-b_{020}\cos(4\theta) + b_{020} + 3b_{200}\\
& -4c_{020} - 4c_{200} + 4b_{200}\cos(2\theta) + 4c_{020}\cos(2\theta) - 4c_{200}\cos(2\theta)\\
& +b_{200}\cos(4\theta) + 2b_{110}\sin(2\theta) - 4c_{110}\sin(2\theta) + b_{110}\sin(4\theta))\Big)\\
& +R\Big(\frac{1}{4}(3b_{100}\cos\theta - 4c_{100}\cos\theta + b_{100}\cos(3\theta) + b_{010}\sin\theta - 4c_{010}\sin\theta\\
& +b_{010}\sin(3\theta))Z^2 + (-b_{101}\cos\theta - b_{011}\sin\theta)Z\Big) + \lambda_2 Z + \frac{1}{2}Z^3(b_{000}\cos(2\theta)\\
& +b_{000} - 2c_{000}) + \frac{1}{2}Z^2(-2b_{001} - \lambda_1 - \lambda_1\cos(2\theta)),
\end{aligned}
$$

$$
\begin{aligned}
F_{31} = \;& RZ^3\sin\theta\cos^5\theta,\\
F_{32} = \;& \frac{1}{8}\Big((-4\cos(2\theta)b_{000} - \cos(4\theta)b_{000} - 3b_{000} + 4c_{000} + 4c_{000}\cos(2\theta))Z^4\\
& +(4\cos(2\theta)b_{001} + 4b_{001} - 8c_{001} + 3\lambda_1 + 4\lambda_1\cos(2\theta) + \lambda_1\cos(4\theta))Z^3\\
& -4(2b_{002} + \lambda_2 + \lambda_2\cos(2\theta))Z^2 + 8\lambda_3 Z + R\Big(\frac{1}{2}(-10b_{100}\cos\theta + 12c_{100}\cos\theta\\
& -5b_{100}\cos(3\theta) + 4c_{100}\cos(3\theta) - b_{100}\cos(5\theta) - 2b_{010}\sin\theta + 4c_{010}\sin\theta\\
& -3b_{010}\sin(3\theta) + 4c_{010}\sin(3\theta) - b_{010}\sin(5\theta))Z^3 + 2(3b_{101}\cos\theta - 4c_{101}\cos\theta\\
& +b_{101}\cos(3\theta) + b_{011}\sin\theta - 4c_{011}\sin\theta + b_{011}\sin(3\theta))Z^2 - 8(b_{102}\cos\theta\\
& +b_{012}\sin\theta)Z\Big) + R^3\Big(\frac{1}{2}z(2c_{120}\cos\theta + 10c_{300}\cos\theta - c_{120}\cos(3\theta)\\
& +5c_{300}\cos(3\theta) - c_{120}\cos(5\theta) + c_{300}\cos(5\theta) + 2c_{030}\sin\theta + 2c_{210}\sin\theta\\
& +c_{030}\sin(3\theta) + 3c_{210}\sin(3\theta) - c_{030}\sin(5\theta) + c_{210}\sin(5\theta)) - 2(c_{121}\cos\theta\\
& +3c_{301}\cos\theta - c_{121}\cos(3\theta) + c_{301}\cos(3\theta) + 3c_{031}\sin\theta + c_{211}\sin\theta\\
& -c_{031}\sin(3\theta) + c_{211}\sin(3\theta))\Big) + R^2\Big(\frac{1}{4}(-b_{020}\cos(2\theta) + 2b_{020}\cos(4\theta)\\
& +b_{020}\cos(6\theta) - 2b_{020} - 10b_{200} + 4c_{020} + 12c_{200} - 15b_{200}\cos(2\theta)\\
& +16c_{200}\cos(2\theta) - 6b_{200}\cos(4\theta) - 4c_{020}\cos(4\theta) + 4c_{200}\cos(4\theta)\\
& -b_{200}\cos(6\theta) - 5b_{110}\sin(2\theta) + 8c_{110}\sin(2\theta) - 4b_{110}\sin(4\theta) + 4c_{110}\sin(4\theta)\\
& -b_{110}\sin(6\theta))Z^2 + (-\cos(4\theta)b_{021} + b_{021} + 3b_{201} - 4c_{021} - 4c_{201}
\end{aligned}
$$

$$
\begin{aligned}
&+4b_{201}\cos(2\theta)+4c_{021}\cos(2\theta)-4c_{201}\cos(2\theta)+b_{201}\cos(4\theta)+2b_{111}\sin(2\theta)\\
&-4c_{111}\sin(2\theta)+b_{111}\sin(4\theta))Z+4b_{022}(\cos(2\theta)-b_{022}-b_{202}-b_{202}\cos(2\theta)\\
&-b_{112}\sin(2\theta))\Big)\Big),\\
F_{41}=&\ -RZ^4\sin\theta\cos^7\theta,\\
F_{42}=&\ \frac{1}{32}(15\cos(2\theta)b_{000}+6b_{000}\cos(4\theta)+b_{000}\cos(6\theta)+10b_{000}-12c_{000}\\
&-16c_{000}\cos(2\theta)-4c_{000}\cos(4\theta))Z^5+\frac{1}{32}(-16b_{001}\cos(2\theta)-4b_{001}\cos(4\theta)\\
&-12b_{001}+16c_{001}-10\lambda_1+16c_{001}\cos(2\theta)-15\lambda_1\cos(2\theta)-6\lambda_1\cos(4\theta)\\
&-\lambda_1\cos(6\theta))Z^4+\frac{1}{8}(4\cos(2\theta)b_{002}+4b_{002}-8c_{002}+3\lambda_2+4\lambda_2\cos(2\theta)\\
&+\lambda_2\cos(4\theta))Z^3+\tfrac{1}{2}(-2b_{003}-\lambda_3-\lambda_3\cos(2\theta))Z^2+\lambda_4 Z+R\Big(\frac{1}{64}(35b_{100}\cos\theta\\
&-40c_{100}\cos\theta+21b_{100}\cos(3\theta)-20c_{100}\cos(3\theta)+7b_{100}\cos(5\theta)\\
&-4c_{100}\cos(5\theta)+b_{100}\cos(7\theta)+5b_{010}\sin\theta-8c_{010}\sin\theta+9b_{010}\sin(3\theta)\\
&-12c_{010}\sin(3\theta)+5b_{010}\sin(5\theta)-4c_{010}\sin(5\theta)+b_{010}\sin(7\theta))Z^4\\
&+\frac{1}{16}(-10b_{101}\cos\theta+12c_{101}\cos\theta-5b_{101}\cos(3\theta)+4c_{101}\cos(3\theta)\\
&-b_{101}\cos(5\theta)-2b_{011}\sin\theta+4c_{011}\sin\theta-3b_{011}\sin(3\theta)+4c_{011}\sin(3\theta)\\
&-b_{011}\sin(5\theta))Z^3+\frac{1}{4}(3b_{102}\cos\theta-4c_{102}\cos\theta+b_{102}\cos(3\theta)+b_{012}\sin\theta\\
&-4c_{012}\sin\theta+b_{012}\sin(3\theta))Z^2+(-b_{103}\cos\theta-b_{013}\sin\theta)Z\Big)\\
&+R^3\Big(\frac{1}{64}(-5c_{120}\cos\theta-35c_{300}\cos\theta+c_{120}\cos(3\theta)-21c_{300}\cos(3\theta)\\
&+3c_{120}\cos(5\theta)-7c_{300}\cos(5\theta)+c_{120}\cos(7\theta)-c_{300}\cos(7\theta)-3c_{030}\sin\theta\\
&-5c_{210}\sin\theta-3c_{030}\sin(3\theta)-9c_{210}\sin(3\theta)+c_{030}\sin(5\theta)-5c_{210}\sin(5\theta)\\
&+c_{030}\sin(7\theta)-c_{210}\sin(7\theta))Z^2+\frac{1}{16}(2c_{121}\cos\theta+10c_{301}\cos\theta-c_{121}\cos(3\theta)\\
&+5c_{301}\cos(3\theta)-c_{121}\cos(5\theta)+c_{301}\cos(5\theta)+2c_{031}\sin\theta+2c_{211}\sin\theta\\
&+c_{031}\sin(3\theta)+3c_{211}\sin(3\theta)-c_{031}\sin(5\theta)+c_{211}\sin(5\theta))Z+\frac{1}{4}(-c_{122}\cos\theta\\
&-3c_{302}\cos\theta+c_{122}\cos(3\theta)-c_{302}\cos(3\theta)-3c_{032}\sin(\theta)-c_{212}\sin\theta\\
&+c_{032}\sin(3\theta)-c_{212}\sin(3\theta))\Big)+R^2\Big(\frac{1}{128}(4b_{020}\cos(2\theta)-4b_{020}\cos(4\theta)\\
&-4b_{020}\cos(6\theta)-b_{020}\cos(8\theta)+5b_{020}+35b_{200}-8c_{020}-40c_{200}\\
&+56b_{200}\cos(2\theta)-4c_{020}\cos(2\theta)-60c_{200}\cos(2\theta)+28b_{200}\cos(4\theta)\\
&+8c_{020}\cos(4\theta)-24c_{200}\cos(4\theta)+8b_{200}\cos(6\theta)+4c_{020}\cos(6\theta)\\
&-4c_{200}\cos(6\theta)+b_{200}\cos(8\theta)+14b_{110}\sin(2\theta)-20c_{110}\sin(2\theta)\\
&+14b_{110}\sin(4\theta)-16c_{110}\sin(4\theta)+6b_{110}\sin(6\theta)-4c_{110}\sin(6\theta)\\
&+b_{110}\sin(8\theta))Z^3+\frac{1}{32}(-b_{021}\cos(2\theta)+2b_{021}\cos(4\theta)+b_{021}\cos(6\theta)-2b_{021}\\
&-10b_{201}+4c_{021}+12c_{201}-15b_{201}\cos(2\theta)+16c_{201}\cos(2\theta)-6b_{201}\cos(4\theta)\\
&-4c_{021}\cos(4\theta)+4c_{201}\cos(4\theta)-b_{201}\cos(6\theta)-5b_{111}\sin(2\theta)
\end{aligned}
$$

$$+8c_{111}\sin(2\theta) - 4b_{111}\sin(4\theta) + 4c_{111}\sin(4\theta) - b_{111}\sin(6\theta))Z^2$$
$$+\frac{1}{8}(-\cos(4\theta)b_{022} + b_{022} + 3b_{202} - 4c_{022} - 4c_{202} + 4b_{202}\cos(2\theta)$$
$$+4c_{022}\cos(2\theta) - 4c_{202}\cos(2\theta) + b_{202}\cos(4\theta) + 2b_{112}\sin(2\theta) - 4c_{112}\sin(2\theta)$$
$$+b_{112}\sin(4\theta))Z + \frac{1}{2}(\cos(2\theta)b_{023} - b_{023} - b_{203} - b_{203}\cos(2\theta) - b_{113}\sin(2\theta))\Big).$$

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