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Liouvillian and Analytic Integrability of the Quadratic Vector Fields Having an Invariant Ellipse

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Abstract We characterize the Liouvillian and analytic integrability of the quadratic polynomial vector fields in $\mathbb{R}^2$ having an invariant ellipse. More precisely, a quadratic system having an invariant ellipse can be written into the form $\dot{x} = x^2 + y^2 - 1 + y(ax + by + c)$, $\dot{y} = -x(ax + by + c)$, and the ellipse becomes $x^2 + y^2 = 1$. We prove that

(i) this quadratic system is analytic integrable if and only if $a = 0$;

(ii) if $x^2 + y^2 = 1$ is a periodic orbit, then this quadratic system is Liouvillian integrable if and only if $x^2 + y^2 = 1$ is not a limit cycle; and

(iii) if $x^2 + y^2 = 1$ is not a periodic orbit, then this quadratic system is Liouvilian integrable if and only if $a = 0$.

Keywords Liouvillian integrability, quadratic planar polynomial vector fields, invariant ellipse

MR(2010) Subject Classification 34C05, 34A34, 34C14

1 Introduction and Statement of Main Results

We study polynomial differential systems in $\mathbb{R}^2$ defined by

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

where P and Q are polynomials with real coefficients such that the maximum degree of P and Q is at most m. When $m = 2$, we call these differential systems simply *quadratic systems*. The dot denotes derivative with respect to the independent variable t, which is called here the *time*. Associated with system (1.1), we have the quadratic polynomial vector field $\mathcal{X}$ with

$$\mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

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