PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 10, October 2014, Pages 3533–3543
S 0002-9939(2014)12085-5
Article electronically published on June 23, 2014

ALGEBRAIC INVARIANT CURVES AND FIRST INTEGRALS FOR RICCATI POLYNOMIAL DIFFERENTIAL SYSTEMS

JAUME LLIBRE AND CLÀUDIA VALLS

(Communicated by Yingfei Yi)

ABSTRACT. We study the algebraic invariant curves and first integrals for the Riccati polynomial differential systems of the form $x' = 1$, $y' = a(x)y^2 + b(x)y + c(x)$, where $a(x)$, $b(x)$ and $c(x)$ are polynomials. We characterize them when $c(x) = \kappa(b(x) - \kappa a(x))$ for some $\kappa \in \mathbb{C}$. We conjecture that algebraic invariant curves and first integrals for these Riccati polynomial differential systems only exist if $c(x) = \kappa(b(x) - \kappa a(x))$ for some $\kappa \in \mathbb{C}$.

1. Introduction and statement of the main results

One of the more classical problems in the qualitative theory of planar differential equations depending on parameters is to characterize the existence or not of first integrals in the function of these parameters.

Let x and y be complex variables. We consider the system

$$\dot{x} = 1, \qquad \dot{y} = a(x)y^2 + b(x)y + c(x), \tag{1}$$

where $a(x)$, $b(x)$ and $c(x)$ are C^1 functions on x and the prime denotes a derivative with respect to the time t that can be either real or complex. In fact, if $a(x)c(x) \not\equiv 0$ these systems are called *Riccati differential systems*, if $a(x) \not\equiv 0$ and $c(x) \equiv 0$ they are a particular case of *Bernouilli differential systems*, and if $a(x) \equiv 0$, then they are *linear differential systems*.

Differential systems (1) are named after Count Jacobo Francesco Riccati (1676–1754). These equations have been intensively studied, and hundreds of applications have been found. Thus looking at MathSciNet there are more than 4000 papers having in their title the words "Riccati equations". These equations have been studied in many books; see for instance [7–9, 13, 15, 19]. They are important since they can be used to solve second-order ordinary differential equations. An important application of the Riccati differential systems is to the 3rd order Schwarzian differential equation which appears in the theory of conformal mapping and univalent functions; see [14] for more details and references.

Our interest is on the *Riccati polynomial differential systems* (1), i.e. when the functions $a(x)$, $b(x)$ and $c(x)$ are polynomials. More precisely, we want to study

Received by the editors November 18, 2011 and, in revised form, December 13, 2011; May 1, 2012; September 6, 2012; and October 28, 2012.

2010 *Mathematics Subject Classification.* Primary 34C05, 34A34, 34C14.

Key words and phrases. Algebraic first integrals, algebraic invariant curves, Riccati polynomial differential equations.

The first author was partially supported by the MICINN/FEDER grant MTM2008–03437, AGAUR grant 2009SGR-410, ICREA Academia and FP7-PEOPLE-2012-IRSES-316338.

The second author was partially supported by the FCT through CAMGDS, Lisbon.