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On the Darboux integrability of the Painlevé II equations

Jaume Llibre
Departament de Matemàtiques, Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Catalonia, Spain
jllibre@mat.uab.cat

Claudia Valls
Departamento de Matemática, Instituto Superior Técnico
1049–001 Lisboa, Portugal
cvalls@math.ist.utl.pt

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In this paper we prove the non–existence of Darboux first integrals for the Painlevé II equations

$$\dot{x} = y - \frac{z}{2} - x^2, \quad \dot{y} = \alpha + \frac{1}{2} + 2xy, \quad \dot{z} = 1$$

for all values of $\alpha \in \mathbb{C} \setminus \{\alpha_n : n = 2, 4, \ldots\}$. These α_n are real and larger than $-1/2$.

Keywords: Painlevé transcendents; Hamiltonian systems; Darboux integrability.

1. Introduction and statement of the main results

The Painlevé equations are Hamiltonian systems that depend on parameters and whose solutions give rise to the so-called Painlevé transcendents. The Painlevé transcendents are solutions to certain nonlinear second-order ordinary differential equations in the complex plane whose only movable singularities are ordinary poles and which cannot be integrated in terms of other known functions or transcendents.

In this paper we study the Darboux integrability of the Painlevé II equations which can be written in the form (see [2–5] for details):

$$\dot{x} = y - \frac{z}{2} - x^2, \quad \dot{y} = \alpha + \frac{1}{2} + 2xy, \quad \dot{z} = 1, \quad \dot{w} = \frac{y}{2}$$

with the Hamiltonian

$$H = \frac{1}{2}y^2 - \left(x^2 + \frac{z}{2}\right)y - \left(\alpha + \frac{1}{2}\right)x + w.$$

Here the parameter α is complex. Using H we can eliminate the variable w making

$$w = H - \frac{1}{2}y^2 + \left(x^2 + \frac{z}{2}\right)y + \left(\alpha + \frac{1}{2}\right)x,$$

and study the integrability of the system

$$\dot{x} = y - \frac{z}{2} - x^2, \quad \dot{y} = \alpha + \frac{1}{2} + 2xy, \quad \dot{z} = 1. \tag{1.1}$$

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