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Analytic integrability of Hamiltonian systems with exceptional potentials

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ABSTRACT

We study the existence of analytic first integrals of the complex Hamiltonian systems of the form

$$H=\frac{1}{2}\sum_{i=1}^{2}p_i^2+V_l(q_1,q_2)$$

with the homogeneous polynomial potential

$$V_l(q_1,q_2)=\alpha(q_2-iq_1)^l(q_2+iq_1)^{k-l},\quad l=0,\dots,k,\quad \alpha\in\mathbb{C}\setminus\{0\}$$

of degree k called *exceptional potentials*. In Remark 2.1 of Ref. [7] the authors state: *The exceptional potentials V_0, V_1, V_{k-1}, V_k and $V_{k/2}$ when k is even are integrable with a second polynomial first integral. However nothing is known about the integrability of the remaining exceptional potentials.* Here we prove that the exceptional potentials with k even different from V_0, V_1, V_{k-1}, V_k and $V_{k/2}$, have no independent analytic first integral different from the Hamiltonian one.

1. Introduction and statement of the main results

Ordinary differential equations in general and Hamiltonian systems in particular play a very important role in many branches of the applied sciences. The question whether a differential system admits a first integral is of fundamental importance as first integrals give conservation laws for the model and that enables to lower the dimension of the system. Moreover, knowing a sufficient number of first integrals allows to solve the system explicitly. Until the end of the 19th century the majority of scientists thought that the equations of classical mechanics were integrable and finding the first integrals was mainly a problem of computation. In fact, now we know that the integrability is a non-generic phenomenon inside the class of Hamiltonian systems (see [8]), and in general it is very hard to determine whether a given Hamiltonian system is integrable or not.

In this work we are concerned with the integrability of the natural Hamiltonian systems defined by a Hamiltonian function of the form

$$H=\frac{1}{2}\sum_{i=1}^{2}p_i^2+V(q_1,q_2), \tag{1}$$

where $V(q_1,q_2)\in\mathbb{C}[q_1,q_2]$ is a homogeneous polynomial potential of degree k. As usual $\mathbb{C}[q_1,q_2]$ is the ring of polynomial functions over $\mathbb{C}$ in the variables q_1 and q_2. To be more precise we consider the following system of four differential equations

$$\dot q_i=p_i,\quad \dot p_i=-\frac{\partial V}{\partial q_i},\qquad i=1,2. \tag{2}$$

Let $A=A(\mathbf{q},\mathbf{p})$ and $B=B(\mathbf{q},\mathbf{p})$ be two functions, where $\mathbf{p}=(p_1,p_2)$ and $\mathbf{q}=(q_1,q_2)$. We define the *Poisson bracket* of A and B as

$$\{A,B\}=\sum_{i=1}^{2}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right).$$

The functions A and B are *in involution* if $\{A,B\}=0$. A non-constant function $F=F(\mathbf{q},\mathbf{p})$ is a *first integral* for the Hamiltonian system (2) if it is in involution with the Hamiltonian function H, i.e. $\{H,F\}=0$. Since the Poisson bracket is antisymmetric it follows that H itself is always a first integral. A 2-degree of freedom

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