DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **21**, Number **2**, March **2016**

doi:10.3934/dcdsb.2016.21.xx

pp. **X–XX**

ON THE ANALYTIC INTEGRABILITY OF THE LIÉNARD ANALYTIC DIFFERENTIAL SYSTEMS

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Abstract. We consider the Liénard analytic differential systems $\dot{x} = y$, $\dot{y} = -g(x) - f(x)y$, where $f, g : \mathbb{R} \to \mathbb{R}$ are analytic functions and the origin is an isolated singular point. Then for such systems we characterize the existence of local analytic first integrals in a neighborhood of the origin and the existence of global analytic first integrals.

1. **Introduction and statement of the main results.** One of the more classical problems in the qualitative theory of planar analytic differential systems in $\mathbb{R}^2$ is to characterize the existence of analytic first integrals in a neighborhood of an isolated singular point, and in particular the existence of a global analytic first integral when the differential system is defined in the whole $\mathbb{R}^2$.

One of the best and oldest results in this direction is the analytic nondegerate center theorem. In order to be more precise we recall some definitions. A singular point is a *nondegenerate center* if it is a center with eigenvalues purely imaginary. If a real planar analytic system has a nondegenerate center at the origin, then after a linear change of variables and a rescaling of the time variable, it can be written in the form:

$$\begin{aligned} \dot{x} &= y + X(x, y), \\ \dot{y} &= -x + Y(x, y), \end{aligned} \tag{1}$$

where $X(x, y)$ and $Y(x, y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin.

Let U be an open subset of $\mathbb{R}^2$, $H : U \to \mathbb{R}$ be a nonconstant analytic function and $\mathcal{X}$ be an analytic vector field defined on U. Then H is an *analytic first integral* of $\mathcal{X}$ if H is constant on the solutions of $\mathcal{X}$; i.e. if $\mathcal{X}H = 0$.

The next result is due to Poincaré [9] and Liapunov [6], see also Moussu [8].

2010 *Mathematics Subject Classification.* Primary: 34C05, 34A34, 34C14.

Key words and phrases. Formal integrability, analytic integrability, Liénard system.

We thank to the reviewers their comments and suggestions which help us to improve this paper.

The first author is partially supported by a MINECO grant MTM2013-40998-P, an AGAUR grant number 2014-SGR568, the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, and by a MINECO/FEDER grant UNAB13-4E-1604.

The second author is supported by Portuguese National Funds through FCT - Fundação para a Ciència e a Tecnologia trough UID/MAT/04459/2013.

Theorem 1 (Analytic nondegerate center theorem)**.** *The analytic differential system (1) has a nondegerate center at the origin if and only if there exists an analytic first integral defined in a neighborhood of the origin.*

One of the more studied differential systems are the so–called *generalized Liénard equation*

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{2}$$

which were studied by many researchers. Such dynamical systems appear very often in several branches of the sciences, such as biology, chemistry, mechanics, electronics, etc. The differential equation (2) of second order can be written as the equivalent 2–dimensional Liénard differential system of first order

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{3}$$

When $g(x) = x$ the Liénard differential systems (3) are called the *classical Liénard systems*. The main objective of this paper is to study the analytical integrability of the Liénard systems (3) depending on the analytic functions f and g.

In order that the origin of system (3) be a singular point we need that $g(0) = 0$, and since it must be isolated we need that $g(x) \not\equiv 0$. Therefore, since we want to study the local analytic integrability at the isolated singular point located at the origin in the rest of the paper *we always assume that*

$$g(0) = 0, \quad g(x) \not\equiv 0 \quad \text{and} \quad f(x) \not\equiv 0. \tag{4}$$

Of course if $g(x) \equiv 0$ or $f(x) \equiv 0$ then the Liénard differential system becomes a differential equation with separable variables.

The functions

$$F(x) = \int_0^x f(s)ds \quad \text{and} \quad G(x) = \int_0^x g(s)ds$$

are useful in the study of the Liénard system (3).

Through the paper $\mathbb{Z}^+$ will denote the set of non-negative integers, $\mathbb{Z}^-$ will denote the set of negative integers, $\mathbb{Q}^+$ will denote the set of non-negative rational numbers and $\mathbb{Q}^-$ will denote the set of negative rational numbers.

In the spacial case in which $g(x) = -\dfrac{pq}{(p-q)^2} f(x)F(x)$ with $p, q \in \mathbb{Z}^+ \setminus \{0\}$ and $p \neq q$ we have that

$$H = ((p-q)y + pF(x))^p ((p-q)y - qF(x))^q \tag{5}$$

is a global analytic first integral of system (3).

Due to technicalities we will also *assume* in the paper that

(H0) if $f(0) \neq 0$ then $g'(0) \neq -\frac{pq}{(p-q)^2} f(0)^2$ for some $p, q \in \mathbb{Z}^+ \setminus \{0\}$ and $p \neq q$.

Note that when $f(0) \neq 0$, $g'(0) = -\frac{pq}{(p-q)^2} f(0)^2$ and $g(x) = -\frac{pq}{(p-q)^2} f(x)F(x)$ then system (3) is integrable with first integral given in (5) but we are not considering this case in our paper.

When $f(0) = g(0) = g'(0) = 0$ then using the Taylor expansion for the analytic functions $g(x)$ and $f(x)$ we can write:

$$g(x) = \sum_{i \geq 0} g_{\ell_0+i} x^{\ell_0+i}, \quad f(x) = \sum_{i \geq 0} f_{\ell_1+i} x^{\ell_1+i},$$

where $\ell_0 \geq 2$ and $\ell_1 \geq 1$. Now depending on ℓ_0 and ℓ_1 we will consider other assumptions on our systems. We will assume that:

(H1) either $\ell_0 = \ell_1 + 1$;

(H2) or $\ell_0 > 2\ell_1 + 1$.

The main result of this paper is the following.

Theorem 2. *System (3) under assumptions (4) and satisfying hypothesis H0 has a local analytic first integral in a neighborhood of the origin if and only if one of the following conditions hold.*

(a.1) $f(0) = g(0) = 0$, $g'(0) > 0$ *and* $F(x) = \Phi(G(x))$ *for some analytic function* $\Phi(x)$ *with* $\Phi(0) = 0$*;*

(a.2) $f(0) = g(0) = 0$, $g'(0) < 0$ *and* $F(x) = \sqrt{2|G(x)|}\tilde{\Phi}(2|G(x)|)$ *for some analytic function* $\tilde{\Phi}(x)$.

Note that when $\ell_0 = \ell_1$ then we can have an analytic first integral as the following example shows.

Example 3. We consider the system

$$\dot{x} = y, \quad \dot{y} = x^{\ell_0}(a_0 + a_1 y). \tag{6}$$

System (6) satisfies $\ell_0 = \ell_1$ and it has a first integral of the form

$$H = (a_0 + a_1 y)^{a_0(1+\ell_0)} e^{a_1(a_1 x^{1+\ell_0} - (1+\ell_0)y)}$$

that is globally analytic when $a_0 \in \mathbb{Z}$ (taking H^{-1} when $a_0 \in \mathbb{N}^-$).

As a Corollary of Theorem 2 we obtain the following characterization of the analytic first integrals of the so–called *classical* Liénard equations.

Theorem 4. *System (3) with* $g(x) = x$ *and* $f(x) \not\equiv 0$ *has a local analytic first integral in a neighborhood of the origin if and only if* $F(x) = F(-x)$.

In the classical Liénard system, i.e., when $g(x) = x$ and $f(x) \not\equiv 0$, we have that (4) and hypotheses H0–H2 automatically hold. Therefore we are on the assumptions of Theorem 2. Furthermore, $g'(0) = 1 > 0$ and then only statement (a.1) can hold. In this case, since $G(x) = x^2/2$ we have that system (3) has a local analytic first integral in a neighborhood of the origin if and only if $F(x) = \Phi(x^2/2)$, for some analytic function Φ with $\Phi(0) = 0$. This clearly implies Theorem 4.

The paper has been divided as follows. In Section 2 we introduce some preliminary results and as a corollary as those results we prove Theorem 2(a.1). In Section 3 we prove Theorem 2 when $f(0) \neq 0$ and hypothesis H0 holds. In Section 4 we prove Theorem 2 when $f(0) = g(0) = 0$ and $g'(0) < 0$. In Sections 5 we prove Theorem 2 when hypothesis H1 holds, and in finally in Section 6 we prove Theorem 2 when hypothesis H2 holds.

2. **Preliminary results.**

In this section we shall introduce two results that will be used through the paper. Let $h = h(x, y)$ be the vector field associated to our system (3).

The following result is due Poincaré and its proof can be found in [11], see also [4].

Theorem 5. *Assume that the eigenvalues* $\lambda_1 \neq 0$ *and* $\lambda_2 \neq 0$ *at some singular point of* h *do not satisfy any resonance condition of the form*

$$\lambda_1 k_1 + \lambda_2 k_2 = 0 \quad \textit{for} \quad k_1, k_2 \in \mathbb{Z}^+ \quad \textit{with} \quad k_1 + k_2 > 0.$$

Then system (3) has no local analytic first integrals.

The following result extends the previous one to the case that one eigenvalues is zero, see Li, Llibre and Zhang [5].

Theorem 6. *Assume that the eigenvalues λ_1 and λ_2 at some singular point p of h satisfy that $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then system (3) has no local analytic first integrals in a neighborhood of the singular p if it is isolated.*

We must mention that the singular points appearing in the statements of Theorems 5 and 6 can be real or complex, but our system (3) is always real.

We have the following nice characterization of the centers at the origin for the Liénard systems (3) due to Christopher [1].

Theorem 7 (Center Theorem for analytic Liénard systems)**.** *Let $f(x)$ and $g(x)$ be analytic functions defined in a neighborhood of zero such that $f(0) = g(0) = 0$ and $g'(0) > 0$. Then the Liénard differential system (3) has a nondegenerate center at the origin if and only if $F(x) = \Phi(G(x))$ for some analytic function $\Phi(x)$, with $\Phi(0) = 0$.*

An immediate consequence of Theorems 1 and 7 is:

Corollary 8. *Let $f(x)$ and $g(x)$ be analytic functions defined in a neighborhood of zero such that $f(0) = g(0) = 0$ and $g'(0) > 0$. Then there exists an analytic first integral defined in a neighborhood of the origin if and only if $F(x) = \Phi(G(x))$ for some analytic function $\Phi(x)$, with $\Phi(0) = 0$.*

Note that Corollary 8 is exactly Theorem 2(a.1).

If a real analytic system has a center at the origin and after a linear change of varibles and a rescaling of the time variable, it can be written in the form

$$\dot{x} = y + X(x, y), \quad \dot{y} = Y(x, y),$$

where $X(x, y)$ and $Y(x, y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin, then it is called a *nilpotent center*.

3. Case $f(0) \neq 0$.

In this section we will prove the following result.

Proposition 9. *Let $f(x)$ and $g(x)$ be analytic functions defined in a neighborhood of the origin satisfying (4) such that $f(0) \neq 0$ and hypothesis H0 holds. Then system (3) has no local analytic first integrals in a neighborhood of zero.*

We separate the proof of Proposition 9 into different lemmas.

Lemma 10. *Let $f(x)$ and $g(x)$ be analytic functions defined in a neighborhood of the origin satisfying (4) and such that $f(0) \neq 0$ and $g'(0) = 0$. Then system (3) has no local analytic first integrals in a neighborhood of zero.*

Proof. We note that system (3) has the singular point $(0, 0)$. If $X = (y, -g(x) - f(x)y)$, then the eigenvalues of $DX(0, 0)$ are

$$\frac{-f(0) \pm |f(0)|}{2}.$$

So one eigenvalue is zero and the other is nonzero.

Since $g(x) \not\equiv 0$ and $g(x)$ is analytic, we have that $(0, 0)$ is an isolated singular point of system (3) and by Theorem 6, system (3) has no local analytic first integrals in a neighborhood of zero. □

Now we can assume that $g'(0) \neq 0$.

Lemma 11. *Let $f(x)$ and $g(x)$ be analytic functions defined in a neighborhood of the origin satisfying (4) and such that $f(0) \neq 0$, $g'(0) \neq 0$ and $f(0)^2/g'(0) \notin \mathbb{Q}^-$. Then system (3) has no local analytic first integrals in a neighborhood of zero.*

Proof. We note that system (3) has the singular point $(0,0)$. If $X = (y, -g(x) + f(x)y)$ then the eigenvalues of $DX(0,0)$ are

$$\lambda_1 = \frac{f(0) + \sqrt{f(0)^2 - 4g'(0)}}{2} \quad \text{and} \quad \lambda_2 = \frac{f(0) - \sqrt{f(0)^2 - 4g'(0)}}{2}.$$

Clearly

$$\lambda_1 + \lambda_2 = f(0) \quad \text{and} \quad \lambda_1\lambda_2 = g'(0). \tag{7}$$

Suppose that there exist positive integers k_1 and k_2 such that $k_1\lambda_1 + k_2\lambda_2 = 0$. Note that by Theorem 5 if such integers do not exist the proposition is proved. Then $\lambda_1 = -\alpha\lambda_2$ with α a positive rational. The two equalities of (7) become

$$f(0) = (1-\alpha)\lambda_2, \quad \text{and} \quad g'(0) = -\alpha\lambda_2^2.$$

Since we have

$$\frac{f(0)^2}{g'(0)} = -\frac{(1-\alpha)^2}{\alpha} \in \mathbb{Q}^-.$$

Note that $\alpha \neq 1$ because $f(0) \neq 0$, and $\alpha \neq 0$ because $g'(0) \neq 0$. Therefore since $f(0)^2/g'(0) \in \mathbb{Q}^-$, we cannot have $k_1\lambda_1 + k_2\lambda_2 = 0$, and consequently the proposition is proved. □

Lemma 12. *Let $f(x)$ and $g(x)$ be analytic functions defined in a neighborhood of the origin satisfying (4) and such that $f(0) \neq 0$, $g'(0) \neq 0$ and $f(0)^2/g'(0) = -\alpha \in \mathbb{Q}^-$ and hypothesis H0 holds, i.e., $\alpha \neq pq/(p-q)^2$ for some $p, q \in \mathbb{Z}^+$, $p \neq q$. Then system (3) has no local analytic first integrals in a neighborhood of zero.*

Proof. We write $g'(0) = -\alpha f(0)^2$ with $\alpha \in \mathbb{Q}^+ \setminus \{0\}$. Doing the rescaling $(X, Y, T) = (f(0)x, y, f(0)t)$ system (3) becomes of the form

$$x' = y, \quad \dot{y} = \alpha x - y + h.o.t.,$$

where we have written again (x, y, t) instead of (X, Y, T) and where h.o.t. means terms of higher order. We assume that $H = H(x, y)$ is a local analytic first integral in a neighborhood of the origin. We write it as $H = \sum_{k \geq 0} H_k(x, y)$ where H_k are homogeneous polynomials of degree k. We will show by induction that

$$H_k = 0 \quad \text{for } k \geq 1 \tag{8}$$

Then clearly from (8) we will obtain that system (3) has no local analytic first integrals, and the proof of the proposition will be done.

Since H is a first integral it must satisfy

$$(\alpha f(0)x - y)\frac{\partial H}{\partial y} + y\frac{\partial H}{\partial x} = 0. \tag{9}$$

Now we will do the induction. The terms of degree one in (9) must satisfy

$$(\alpha x - y)\frac{\partial H_1}{\partial y} + y\frac{\partial H_1}{\partial x} = 0.$$

Suppose that $H_1 = ax + by$. Then the previous equation becomes $(\alpha x - y)b + ay = 0$. So $\alpha b = 0$, since $\alpha \neq 0$ we get that $\partial H_1/\partial y = b = 0$. Then we also obtain $\partial H_1/\partial x = 0$ and thus $H_1 = 0$ which proves (8) for $k = 1$. Now we assume that

(8) is true for $k = 1, \dots, j-1$ with $j \geq 2$ and we will prove it for $k = j$. By the induction hypothesis, the terms of order j in (9) must satisfy

$$(\alpha x - y)\frac{\partial H_j}{\partial y} + y\frac{\partial H_j}{\partial x} = 0.$$

Therefore, either $H_j = 0$ or H_j is a first integral of the linear system

$$\dot{x} = y, \quad \dot{y} = \alpha x - y.$$

Computing a first integral of this system, we obtain that it must be a function of

$$G = \left(\frac{1}{2}(1-\sqrt{1+4\alpha})x + y\right)^{-1+\sqrt{1+4\alpha}} \left(\frac{1}{2}(1+\sqrt{1+4\alpha})x + y\right)^{1+\sqrt{1+4\alpha}}.$$

The unique possibility for a power or function of G to be a polynomial is that

$$-1+\sqrt{1+4\alpha} = \frac{n_1}{n_2} \quad \text{and} \quad 1+\sqrt{1+4\alpha} = \frac{n_3}{n_2}, \quad n_1, n_3 \in \mathbb{Z}^+, n_3 \neq n_1.$$

Then we have that

$$\frac{n_1}{n_2} + 1 = \frac{n_3}{n_2} - 1, \quad \text{that is} \quad n_2 = \frac{n_3 - n_1}{2}.$$

Hence,

$$\sqrt{1+4\alpha} = \frac{n_1 + n_2}{n_2} = \frac{n_1 + n_3}{n_3 - n_1}$$

which yields

$$\alpha = \frac{1}{4}\left(\left(\frac{n_1+n_3}{n_3-n_1}\right)^2 - 1\right) = \frac{n_1 n_3}{(n_3-n_1)^2}, \quad n_1, n_3 \in \mathbb{Z}^+, n_1 \neq n_3,$$

a contradiction. Since all first integrals are functions of G it is clear that H_j cannot be a homogeneous polynomial of degree j. Hence $H_j = 0$ and the induction process has ended. □

From Lemmas 10, 11 and 12 it follows the proposition.

4. **Case** $f(0) = g(0) = 0$**,** $g'(0) < 0$**.** In this section we will prove the following proposition.

Proposition 13. *Let $f(x)$ and $g(x)$ be analytic functions defined in a neighborhood of the origin satisfying (4) such that $f(0) = 0$. Then there exists an analytic first integral defined in a neighborhood of the origin if and only if $F(x) = \Phi(2|G(x)|)$ for some analytic function $\Phi(x)$.*

Proof. Making the transformation $(X, Y) = (x, y + F(x)$, system (3) becomes

$$x' = y - F(x), \quad y' = -g(x) \tag{10}$$

where we have written again (x, y) instead of (X, Y).

Let u be the negative root of $2G$. From the hypothesis on g (that is, $g(0) = 0$ and $g'(0) < 0$) it is clear that it is well defined and analytic in a neighborhood of $x = 0$. Thus

$$u = -(-2G(x))^{1/2}\text{sgn}\,(x) = -(-g'(0))^{1/2}x + O(x^2) \tag{11}$$

defines an invertible analytic transformation in a neighborhood of $x = 0$. Let $x(u)$ denote its inverse. This transformation takes system (10) to the system

$$u' = -\frac{g(x(u))}{u}(y - F(x(u)), \quad y' = -g(x(u)). \tag{12}$$

Since $g(x(u))/u = (-g'(0))^{1/2} + O(u)$ is analytic in a neighborhood of the origin. We can rescale (12) by multiplying the right-hand side by $u/g(x(u))$ which gives

$$u' = y + F(x(u)), \quad y' = u. \tag{13}$$

We write

$$\tilde{F}(u) = F(x(u)) = \sum_{j \geq 2} a_j u^j, \quad a_j \in \mathbb{R}.$$

Then system (13) becomes

$$u' = y + \tilde{F}(u), \quad y' = u. \tag{14}$$

We introduce the change of variables

$$y = ib, \quad u = a, \tag{15}$$

with this change of variables system (14) becomes

$$\frac{db}{dt} = -ia, \quad \frac{da}{dt} = ib + \tilde{F}(a), \tag{16}$$

where

$$\tilde{F}(a) = \sum_{j \geq 2} \tilde{F}_j a^j.$$

It is clear that system (13) is locally integrable around the origin if and only if system (16) is locally integrable around the origin. Moreover, system (16) is locally integrable if and only if $\tilde{F}_j = 0$ for j odd, see for more details the Appendix. More precisely, the coefficients $\tilde{F}_j$ for j odd are the Poincaré–Liapunov constants of the Lienard analytic differential system (16), and in the papers [2, 15] it is proved that the Lienard differential systems have an analytic first integral in the neighborhood of the origin if and only if all the Poincaré–Liapunov constants are zero.

In short, system (13) has a local analytic first integral in a neighborhood of the origin if and only if $\tilde{F}$ is an even function. Now we express this condition in a more geometrical setting. The argument above shows that there is a local analytic first integral if and only if $\tilde{F}(u) = F(x(u)) = \phi(u^2)$ for some analytic function ϕ. However, $u^2 = 2|G(x)|$ and this concludes the proof of the proposition. □

5. **Case H1.** The main result in this section is the following.

Proposition 14. *System (3) with $\ell_0 \geq 2$ and $\ell_1 = \ell_0 - 1$ has no analytic first integrals.*

Proof. System (3) becomes

$$x' = y \quad y' = g_{\ell_0} x^{\ell_0} + f_{\ell_0 - 1} y x^{\ell_0 - 1} + h.o.t., \tag{17}$$

where h.o.t. denote the higher order terms. We claim that (17) has not global analytic first integrals. We note that the proof of the proposition will follow then from the claim. Now we shall prove the claim.

Let $H = H(x, y) = \sum_{k \geq 1} H_k(x, y)$ be a first integral of (17), where H_k is a homogeneous polynomial of degree k. Then H satisfies

$$y \frac{\partial H}{\partial x} + \left(g_{\ell_0} x^{\ell_0} + f_{\ell_0 - 1} y x^{\ell_0 - 1} + h.o.t \right) \frac{\partial H}{\partial y} = 0. \tag{18}$$

Note that the right-hand side of equation (18) has degree at least ℓ_0. Now we will show by induction that for $k \geq 1$,

$$H_{k-1} = 0 \quad \text{and} \quad \frac{\partial H_k}{\partial x} = \cdots = \frac{\partial H_{k+\ell_0-2}}{\partial x} = 0. \tag{19}$$

We have taken the criterium that for $j \geq 0$, $H_{-j} = 0$. If (19) holds, then since all $H_{k-1} = 0$ for $k \geq 1$, and $H = \sum_{k \geq 1} H_k$ it follows that $H = 0$, a contradiction with the fact that H is a global first integral of system (17). Hence the claim will be proved if we prove the induction hypothesis.

Computing the terms in (18) with degree one we obtain that $y \frac{\partial H_1}{\partial x} = 0$ and since $H_0 = 0$ the induction hypothesis is proved for $k = 1$.

Now we assume that (19) is true for $k = 1, \ldots, l$ ($l \geq 1$) and we will prove it for $k = l + 1$. By the induction hypothesis we have

$$H_{l-i} = 0 \quad \text{and} \quad \frac{\partial H_l}{\partial x} = \cdots = \frac{\partial H_{l+\ell_0-2}}{\partial x} = 0$$

for $i = 1, \ldots, l$ that is,

$$H_{l-i} = 0 \quad \text{and} \quad H_{l+j} = a_{l+j} y^{l+j}, \quad a_{l+j} \in \mathbb{R}, \tag{20}$$

for $i = 1, \ldots, l$ and $j = 0, \ldots, \ell_0 - 2$. Computing the degree $l + \ell_0 - 1$ in (18) and using the induction hypothesis, we get that

$$\begin{aligned} y \frac{\partial H_{l+\ell_0-1}}{\partial x} &= -\left(g_{\ell_0} x^{\ell_0} + f_{\ell_0-1} y x^{\ell_0-1}\right) \frac{\partial H_l}{\partial y} \\ &= -l a_l (g_{\ell_0} x^{\ell_0} + f_{\ell_0-1} y x^{\ell_0-1}) y^{l-1}. \end{aligned} \tag{21}$$

If $l = 1$, then (21) becomes

$$y \frac{\partial H_{\ell_0}}{\partial x} = -a_1 (g_{\ell_0} x^{\ell_0} + f_{\ell_0-1} y x^{\ell_0-1}).$$

From this equation since $g_{\ell_0} \neq 0$ we get that $a_1 = 0$ and $\partial H_{\ell_0}/\partial x = 0$, so $H_1 = 0$ and from equation (20) with $l = 1$ we get $H_1 = 0$, $\partial H_2/\partial x = \cdots = \partial H_{\ell_0}/\partial x = 0$. The induction hypothesis is proved for $k = 2$. Now we assume $l \geq 2$.

First we will prove by induction that for $m \geq 1$,

$$H_{l+m(\ell_0-1)} = (-1)^m a_l g_{\ell_0}^{m-1} (g_{\ell_0} C_{l,m} x + f_{\ell_0-1} K_{l,m} y) x^{(\ell_0+1)m-1} y^{l-2m} + O(y^{l-2m+2}), \tag{22}$$

where $C_{l,m}$, $K_{l,m}$ are positive constants depending on l and m and $O(y^{l-2m+2})$ denote the terms of order greater or equal $l - 2m + 2$ in y. Furthermore, for $j = 1, \ldots, \ell_0 - 2$,

$$H_{l+m(\ell_0-1)+j} = O(y^{l-2m}). \tag{23}$$

Note that from (22) and (23) we have that

$$H_{l+m(\ell_0-1)+j} = O(y^{l-2m}), \quad \text{for } j = 0, \ldots, \ell_0 - 2. \tag{24}$$

Equations (22) and (23) will allow us to complete the proof of the induction hypothesis (19).

Since $l \geq 2$, solving (21) we get

$$\begin{aligned} H_{l+\ell_0-1} &= -a_l l y^{l-2} \left(g_{\ell_0} \frac{x^{\ell_0+1}}{\ell_0 + 1} + \frac{f_{\ell_0-1}}{\ell_0} y x^{\ell_0} \right) + a_{l+1} y^{l+1} \\ &= -a_l g_{\ell_0} C_{l,1} y^{l-2} x^{\ell_0+1} - a_l f_{\ell_0-1} K_{l,1} y^{l-1} x^{\ell_0} + O(y^l), \end{aligned}$$

where $C_{l,1} = l/(\ell_0+1)$ and $K_{1,l} = l/\ell_0$. This proves (22) with $m = 1$. Computing the degree $l + \ell_0 - 1 + n$ in (18) with $1 \le n \le \ell_0 - 2$ and using (20) we get that

$$
\begin{aligned}
y\frac{\partial H_{l+\ell_0-1+n}}{\partial x} &= -\sum_{i=0}^{n} g_{\ell_0+i}x^{\ell_0+i}\frac{\partial H_{l+n-i}}{\partial y} - \sum_{i=0}^{n} f_{\ell_0+i-1}x^{\ell_0+i-1}y\frac{\partial H_{l+n-i}}{\partial y}\\
&= -\sum_{i=0}^{n} g_{\ell_0+i}x^{\ell_0+i}(l+n-i)a_{l+n-i}y^{l+n-i-1}\\
&\quad -\sum_{i=0}^{n} f_{\ell_0-1+i}x^{\ell_0+i-1}(l+n-i)a_{l+n-i}y^{l+n-i}\\
&= -la_l(g_{\ell_0+n}x^{\ell_0+n} + f_{\ell_0-1+n}yx^{\ell_0-1+n})y^{l-1}\\
&\quad -(l+1)a_{l+1}g_{\ell_0+n-1}x^{\ell_0+n-1}y^l + O(y^{l+1}),
\end{aligned}
\tag{25}
$$

that is, after simplifying the right-hand side of equation (25) by y and taking integrals in x we get that

$$
H_{l+\ell_0-1+n} = -\frac{l}{\ell_0+n+1}a_l g_{\ell_0+n}x^{\ell_0+n+1}y^{l-2} + O(y^{l-1}) = O(y^{l-2}).
$$

This proves (23) with $m = 1$ and $j = 1, \ldots, \ell_0 - 2$.

Now we assume that (22) is true for $m = 1, \ldots, n-1$ ($n \ge 2$) and we will prove it for $m = n$.

Computing the degree $l + n(\ell_0 - 1)$ in (18) and using the induction hypothesis and (24) we get that

$$
\begin{aligned}
&y\frac{\partial H_{l+n(\ell_0-1)}}{\partial x}\\
&= -(g_{\ell_0}x^{\ell_0} + f_{\ell_0-1}yx^{\ell_0-1})\frac{\partial H_{l+(n-1)(\ell_0-1)}}{\partial y}\\
&\quad -\sum_{r=1}^{n-1}\sum_{j=1}^{\ell_0-2} g_{r\ell_0+j-2(r-1)}x^{r\ell_0+j-2(r-1)}\frac{\partial H_{l+(n-1-r)(\ell_0-1)+\ell_0-1-j}}{\partial y}\\
&\quad -\sum_{r=1}^{n-1}\sum_{j=1}^{\ell_0-2} f_{r\ell_0+j-2(r-1)-1}x^{r\ell_0+j-2(r-1)-1}y\frac{\partial H_{l+(n-1-r)(\ell_0-1)+\ell_0-1-j}}{\partial y}\\
&= -(g_{\ell_0}x^{\ell_0} + f_{\ell_0-1}yx^{\ell_0-1})\frac{\partial H_{l+(n-1)(\ell_0-1)}}{\partial y} + O(y^{l-2(n-1)+1}).
\end{aligned}
$$

Therefore, $y\partial H_{l+n(\ell_0-1)}/\partial x$ is equal to

$$
\begin{aligned}
&-(g_{\ell_0}x^{\ell_0} + f_{\ell_0-1}yx^{\ell_0-1})\\
&\Big((-1)^{n-1}a_l(l-2(n-1))g_{\ell_0}^{n-1}C_{l,n-1,0}x^{(\ell_0+1)(n-1)}y^{l-2(n-1)-1}\\
&\quad + (-1)^{n-1}a_l(l-2(n-1)+1)g_{\ell_0-1}^{n-2}f_{\ell_0-1}K_{l,n-1,0}x^{(\ell_0+1)(n-1)-1}y^{l-2(n-1)}\\
&\quad + O(y^{l-2(n-1)+1})\Big) + O(y^{l-2(n-1)+1})\\
&= (-1)^n a_l(l-2(n-1))g_{\ell_0}^n C_{l,n-1,0}x^{(\ell_0+1)(n-1)+\ell_0}y^{l-2(n-1)-1}
\end{aligned}
$$

$$+ (-1)^n a_l(l-2(n-1)+1) g_{\ell_0}^{n-1} f_{\ell_0-1} K_{l,n-1,0} x^{(\ell_0+1)(n-1)-1+\ell_0} y^{l-2(n-1)}$$
$$+ (-1)^n a_l(l-2(n-1)) g_{\ell_0-1}^{n-1} f_{\ell_0-1} C_{l,n-1,0} x^{(\ell_0+1)(n-1)+\ell_0-1} y^{l-2(n-1)}$$
$$+ O(y^{l-2(n-1)+1}).$$

This yields that

$$\begin{aligned}\frac{\partial H_{l+n(\ell_0-1)}}{\partial x} &= (-1)^n a_l(l-2(n-1)) g_{\ell_0}^n C_{l,n-1} x^{(\ell_0+1)n-1} y^{l-2(n-1)-2}\\ &+ (-1)^n a_l(l-2(n-1)+1) g_{\ell_0}^{n-1} f_{\ell_0-1} K_{l,n-1} x^{(\ell_0+1)n-2} y^{l-2(n-1)-1}\\ &+ (-1)^n a_l(l-2(n-1)) g_{\ell_0-1}^{n-1} f_{\ell_0-1} C_{l,n-1} x^{(\ell_0+1)n-2} y^{l-2(n-1)-1}\\ &+ O(y^{l-2(n-1)})\end{aligned}$$

and thus

$$\begin{aligned}H_{l+n(\ell_0-1)} &= (-1)^n a_l g_{\ell_0}^n C_{l,n} x^{(\ell_0+1)n} y^{l-2n}\\ &+ (-1)^n a_l g_{\ell_0}^{n-1} f_{\ell_0-1} K_{l,n} x^{(\ell_0+1)n-1} y^{l-2n+1}\\ &+ O(y^{l-2n+2})\end{aligned}$$

with

$$C_{l,n} = \frac{l-2(n-1)}{n(\ell_0+1)} C_{l,n-1},$$

and

$$K_{l,n} = \frac{(l-2(n-1)+1)K_{l,n-1} + (l-2(n-1))C_{l,n-1}}{n(\ell_0+1)-1}.$$

Therefore (22) is proved if $m = n$. Now we want to prove (23) for $m = n$ and $j = 1, \ldots, \ell_0-2$. Computing the degree $l+n(\ell_0-1)+s$ in (18), with $1 \le s \le \ell_0-2$ using the induction hypothesis (together with (22) with $m = n$) and taking into account that as we did before, the terms $H_{l+r(\ell_0-1)+j}$ with $r \le n-2$ and $j = 0, \ldots, \ell_0-2$ are of order $O(y^{l-2r}) = O(y^{l-2n+4})$ (see (24)) that lead to higher order terms, we get that only the terms $H_{l+r(\ell_0-1)+j}$ with $r = n-1$ matter. Hence,

$$\begin{aligned} & y\frac{\partial H_{l+n(\ell_0-1)+s}}{\partial x}\\ &= -\sum_{i=0}^{s} g_{\ell_0+i} x^{\ell_0+i} \frac{\partial H_{l+(n-1)(\ell_0-1)+s-i}}{\partial y}\\ &\quad - \sum_{i=0}^{s} f_{\ell_0+i-1} y x^{\ell_0+i-1} \frac{\partial H_{l+(n-1)(\ell_0-1)+s-i}}{\partial y}\\ &= -\sum_{i=0}^{s} g_{\ell_0+i} x^{\ell_0+i} O(y^{l-2n+1}) - \sum_{i=0}^{s} f_{\ell_0+i-1} x^{\ell_0+i} O(y^{l-2n+2})\\ &= O(y^{l-2n+1}).\end{aligned} \tag{26}$$

Solving (26) we clearly obtain

$$\frac{\partial H_{l+n(\ell_0-1)+s}}{\partial x} = O(y^{l-2n}) \quad \text{that is} \quad H_{l+n(\ell_0-1)+s} = O(y^{l-2n})$$

and concludes the proof of (23).

Now we continue with the proof of the induction hypothesis (20). In fact we shall prove that $a_l = 0$ for $l \ge 2$. Then $H_l = 0$ and from (21) we have that $\partial H_{l+\ell_0-2}/\partial x = 0$. This proves (20). We distinguish two cases.

We first assume that l is odd. Then by (22) with $m = (l-1)/2$ we obtain that

$$H_{l+(l-1)(\ell_0-1)/2} = (-1)^{(l-1)/2} a_l g_{\ell_0}^{(l-1)/2} C_{l,(l-1)/2} x^{(\ell_0+1)(l-1)/2} y + O(y^2).$$

Then computing in (18) the term of degree $l + (l+1)(\ell_0 - 1)/2$ and using (24) we get

$$\begin{aligned} & y \frac{\partial H_{l+(l+1)(\ell_0-1)/2}}{\partial x} \\ & = -(g_{\ell_0} x^{\ell_0} + f_{\ell_0-1} y x^{\ell_0-1})[(-1)^{(l-1)/2} a_l g_{\ell_0}^{(l-1)/2} C_{l,(l-1)/2} x^{(\ell_0+1)(l-1)/2} + O(y)] \\ & \quad + O(y) \\ & = (-1)^{(l+1)/2} a_l g_{\ell_0}^{(l+1)/2} C_{l,(l-1)/2} x^{(\ell_0+1)(l-1)/2+\ell_0} + O(y). \end{aligned} \tag{27}$$

Now setting $y = 0$ in (27) and since $g_{\ell_0} \neq 0$ we get that $a_l = 0$.

Finally, if l is even, then (22) with $m = l/2$ yields

$$\begin{aligned} H_{l+l(\ell_0-1)/2} = & (-1)^{l/2} a_l g_{\ell_0}^{l/2} C_{l,l/2} x^{(\ell_0+1)l/2} \\ & + (-1)^{l/2} a_l g_{\ell_0}^{l/2-1} f_{\ell_0-1} K_{l,l/2} x^{(\ell_0+1)l/2-1} y + O(y^2). \end{aligned}$$

Then computing in (18) the term of degree $l + (l+2)(\ell_0 - 1)/2$ and using (24) we get

$$\begin{aligned} & y \frac{\partial H_{l+(l+2)(\ell_0-1)/2}}{\partial x} \\ & = -(g_{\ell_0} x^{\ell_0} + f_{\ell_0-1} y x^{\ell_0-1})[(-1)^{l/2} a_l g_{\ell_0}^{l/2-1} f_{\ell_0-1} K_{l,l/2} x^{(\ell_0+1)l/2-1} + O(y)] \\ & \quad + O(y) \\ & = (-1)^{(l+1)/2} a_l g_{\ell_0}^{l/2} f_{\ell_0-1} K_{l,l/2} x^{(\ell_0+1)l/2+\ell_0-1} + O(y). \end{aligned} \tag{28}$$

Now setting $y = 0$ in (28) and since $g_{\ell_0} f_{\ell_0-1} \neq 0$ we get that $a_l = 0$. This concludes the proof of the proposition. □

6. **Case H2.** The main result in this section is the following.

Proposition 15. *System (3) with $\ell_0 \geq 2\ell_1 + 2$ has no analytic first integrals.*

Proof. System (3) becomes

$$\begin{aligned} x' &= y + F(x) = y + \sum_{i\geq 0} \frac{f_{\ell_1+i}}{\ell_1+i+1} x^{\ell_1+i+1} = y + \sum_{i\geq 0} \bar{f}_{\ell_1+i} x^{\ell_1+i+1}, \\ y' &= -\sum_{i\geq 0} g_{\ell_0+i} x^{\ell_0+i}, \end{aligned} \tag{29}$$

We claim that (29) has not global analytic first integrals. We note that the proof of the proposition will follow then from the claim. Now we shall prove the claim.

Let $H = H(x,y) = \sum_{k\geq 1} H_k(x,y)$ be a first integral of (29) where H_k is a homogeneous polynomial of degree k. Then H satisfies

$$\big(y + \sum_{i\geq 0} \bar{f}_{\ell_1+i} x^{\ell_1+i+1}\big) \frac{\partial H}{\partial x} - \big(\sum_{i\geq 0} g_{\ell_0+i} x^{\ell_0+i}\big) \frac{\partial H}{\partial y} = 0. \tag{30}$$

Note that the term in (30) before $\partial H/\partial y$ has degree at least ℓ_0. We will show by induction that for $k \geq 1$,

$$H_{k-1} = 0 \quad \text{and} \quad \frac{\partial H_k}{\partial x} = \cdots = \frac{\partial H_{k+\ell_0-2}}{\partial x} = 0. \tag{31}$$

We have taken the criterium that for $j \geq 0$, $H_{-j} = 0$. We note that (31) clearly implies that $H = 0$, a contradiction with the fact that H is a global first integral of system (29). Hence the claim will be proved if we prove the induction hypothesis.

Computing the terms in (30) with degree one we obtain that $y\dfrac{\partial H_1}{\partial x} = 0$ and since $H_0 = 0$ the induction hypothesis is proved for $k = 1$.

Now we assume that (31) is true for $k = 1, \ldots, l$ ($l \geq 1$) and we will prove it for $k = l + 1$. By the induction hypothesis, we have

$$H_{l-i} = 0 \quad \text{and} \quad \frac{\partial H_l}{\partial x} = \cdots = \frac{\partial H_{l+\ell_0-2}}{\partial x} = 0$$

for $i = 1, \ldots, l$ that is,

$$H_{l-i} = 0 \quad \text{and} \quad H_{l+j} = a_{l+j} y^{l+j}, \quad a_{l+j} \in \mathbb{R}, \tag{32}$$

for $i = 1, \ldots, l$ and $j = 0, \ldots, \ell_0 - 2$. Computing the degree $l + \ell_0 - 1$ in (30) and using the induction hypothesis, we get that

$$y\frac{\partial H_{l+\ell_0-1}}{\partial x} = -g_{\ell_0} x^{\ell_0} \frac{\partial H_l}{\partial y} = -a_l l g_{\ell_0} x^{\ell_0} y^{l-1}. \tag{33}$$

If $l = 1$, then (33) becomes

$$y\frac{\partial H_{\ell_0}}{\partial x} = -a_1 g_{\ell_0} x^{\ell_0}.$$

From this equation since $g_{\ell_0} \neq 0$ we get that $a_1 = 0$ and $\partial H_{\ell_0}/\partial x = 0$, so $H_1 = 0$ and from equation (32) with $l = 1$ we get $H_1 = 0$, $\partial H_2/\partial x = \cdots = \partial H_{\ell_0}/\partial x = 0$. The induction hypothesis is proved for $k = 2$. Now we assume $l \geq 2$.

Now show that by the induction hypothesis, we have

$$H_{l+\ell_0-1} = -g_{\ell_0} a_l C_l x^{\ell_0+1} y^{l-2} + O(y^{l+\ell_0-1}), \tag{34}$$

and for $j = 1, \ldots, \ell_1 - 1$,

$$H_{l+\ell_0-1+j} = O(y^{l-2}), \tag{35}$$

where $C_l = l/(\ell_0 + 1)$, $O(y^{l+\ell_0-1})$ denote the terms of order greater than or equal to $l + \ell_0 - 1$ in y and $O(y^{l-2})$ denote the terms of order greater than or equal to $l - 2$ in y. Note that from (34), (35) and since $\ell_0 \geq 2\ell_1 + 2 \geq 4$ ($\ell_1 \geq 1$) we have that $H_{l+\ell_0-1+\kappa} = O(y^{l-2})$ for $\kappa = 0, \ldots, \ell_1 - 1$.

Indeed, since $l \geq 2$, solving (33) we get

$$\begin{aligned} H_{l+\ell_0-1} &= -a_l l y^{l-2} g_{\ell_0} g_{\ell_0} \frac{x^{\ell_0+1}}{\ell_0 + 1} y^{l-2} + a_{l+\ell_0-1} y^{l+\ell_0-1} \\ &= -a_l g_{\ell_0} C_l x^{\ell_0+1} y^{l-2} + O(y^{l+\ell_0-1}), \end{aligned}$$

where $C_l = l/(\ell_0 + 1)$. This proves (34).

Computing the degree $l+\ell_0-1+j$ in (30) with $j=1,\dots,\ell_1-1$ (note that in this case $j-1-i-\ell_1\le -2$) and using (32) we get that

$$\begin{aligned} y\frac{\partial H_{l+\ell_0-1+j}}{\partial x} &= -\sum_{i=0}^{j} g_{\ell_0+i}x^{\ell_0+i}\frac{\partial H_{l+j-i}}{\partial y} \\ &= -\sum_{i=0}^{j} g_{\ell_0+i}x^{\ell_0+i}(l+j-i)a_{l+j-i}y^{l+j-i-1} \\ &= -la_l g_{\ell_0+j}x^{\ell_0+j}y^{l-1}-(l+1)a_{l+1}g_{\ell_0+j-1}x^{\ell_0+j-1}y^{l}+O(y^{l+1}), \end{aligned} \tag{36}$$

that is, after simplifying the right-hand side of equation (36) by y and taking integrals in x we get that

$$H_{l+\ell_0-1+j}=-\frac{l}{\ell_0+j+1}a_l g_{\ell_0+j}x^{\ell_0+j+1}y^{l-2}+O(y^{l-1})=O(y^{l-2}).$$

This proves (35) with $j=1,\dots,\ell_1-1$.

Lemma 16. *For $n=1,\dots,l-2$,*

$$H_{l+\ell_0-1+n\ell_1}=(-1)^{n+1}\bar f_{\ell_1}^{n}a_l g_{\ell_0}K_{l,n}x^{\ell_0+1+n(\ell_1+1)}y^{l-2-n}+O(y^{l-1-n}), \tag{37}$$

and for $j=1,\dots,\ell_1-1$,

$$H_{l+\ell_0-1+n\ell_1+j}=O(y^{l-2-n}), \tag{38}$$

where $K_{l,n}$ is a positive constant depending on l, n; $O(y^{l-1-n})$ denote the terms of order greater than or equal to $l-1-n$ in y and $O(y^{l-2-n})$ denote the terms of order greater than or equal to $l-2-n$ in y.

Proof of the lemma. The proof will be done by induction over n. Computing the degree $l+\ell_0-1+\ell_1$ in (30) and using (32) and (34) we get that

$$\begin{aligned} y\frac{\partial H_{l+\ell_0-1+\ell_1}}{\partial x} &= -\bar f_{\ell_1}x^{\ell_1+1}\frac{\partial H_{l+\ell_0-1}}{\partial x}-\sum_{i=0}^{\ell_1} g_{\ell_0+i}x^{\ell_0+i}\frac{\partial H_{l+\ell_1-i}}{\partial y} \\ &= \bar f_{\ell_1}a_l g_{\ell_0}C_l(\ell_0+1)x^{\ell_0+\ell_1+1}y^{l-2}+O(y^{l}) \\ &\quad -\sum_{i=0}^{\ell_1} g_{\ell_0+i}x^{\ell_0+i}a_{l+\ell_1-i}(l+\ell_1-i)y^{l+\ell_1-i-1} \\ &= \bar f_{\ell_1}a_l g_{\ell_0}C_l(\ell_0+1)x^{\ell_0+\ell_1+1}y^{l-2}+O(y^{l}), \end{aligned}$$

where we have used that $\ell_0\ge 2(\ell_1+1)$. Then, after taking simplifying by y and taking integrals in x we get

$$H_{l+\ell_0-1+\ell_1}=\bar f_{\ell_1}a_l g_{\ell_0}K_{l,1}x^{\ell_0+\ell_1+2}y^{l-3}+O(y^{l-1}),$$

where $K_{l,1}=(\ell_0+1)C_l/(\ell_0+\ell_1+2)$. This proves (37) with $n=1$.

Now computing the degree $l+\ell_0-1+\ell_1+r$ in (30) with $1\le r\le \ell_1-1$ and using (32) we get that

$$\begin{aligned} & y\frac{\partial H_{l+\ell_0-1+\ell_1+r}}{\partial x} \\ &= -\sum_{i=0}^{\ell_0+r-1}\bar f_{\ell_1+i}x^{\ell_1+1+i}\frac{\partial H_{l+\ell_0-1-i+r}}{\partial x}-\sum_{i=0}^{\ell_1+r} g_{\ell_0+i}x^{\ell_0+i}\frac{\partial H_{l+\ell_1+r-i}}{\partial y} \\ &= O(y^{l-2}), \end{aligned} \tag{39}$$

where we have used that the terms in (34) and (35) are of order $O(y^{l-2})$ and that $H_{l+\kappa}$ with $\kappa < \ell_0 - 1$ is also $O(y^{l-2})$. Furthermore, we can write $H_{l+\ell_1+r-1} = H_{l+\ell_0-1+2\ell_1+r-\ell_0+1}$ and since $\ell_0 \geq 2\ell_1+2$ we have that $2\ell_1+r-\ell_0+1 \leq j-1 < \ell_1-1$ and then by equation (35) we have that $H_{l+\ell_1+r-1} = O(y^{l-2})$. Now simplifying (39) by y and integrating in x we conclude that (38) holds for $n = 1$ and $j = 1, \ldots, \ell_1 - 1$.

Now we assume that (37) and (38) hold for $n = 0, \ldots, \tilde{n} - 1$ and we will prove it for $n = \tilde{n}$.

Computing the degree $l + \ell_0 - 1 + \tilde{n}\ell_1$ in (30) and the induction hypothesis, we have

$$
\begin{aligned}
y\frac{\partial H_{l+\ell_0-1+\tilde{n}\ell_1}}{\partial x} &= -\bar{f}_{\ell_1} x^{\ell_1+1} \frac{\partial H_{l+\ell_0-1+(\tilde{n}-1)\ell_1}}{\partial x} \\
&\quad - \sum_{i=1}^{\ell_0-1+(\tilde{n}-1)\ell_1} \bar{f}_{\ell_1+i} x^{\ell_1+1+i} \frac{\partial H_{l+\ell_0-1+(\tilde{n}-1)\ell_1-i}}{\partial x} \\
&\quad - \sum_{i=0}^{\tilde{n}\ell_1} g_{\ell_0+i} x^{\ell_0+i} \frac{\partial H_{l+\tilde{n}\ell_1-i}}{\partial y}.
\end{aligned}
$$

Note that since $i \geq 1$, by (37) and (38) with $n \leq \tilde{n} - 1$ we have

$$
\frac{\partial H_{l+\ell_0-1+(\tilde{n}-1)\ell_1-i}}{\partial x} = \frac{\partial H_{l+\ell_0-1+(\tilde{n}-2)\ell_1+(\ell_1-i)}}{\partial x} = O(y^{l-\tilde{n}}).
$$

Furthermore we can write $l + \tilde{n}\ell_1 - i$ in $\partial H_{l+\tilde{n}\ell_1-i}/\partial y$ of the form

$$
l + \tilde{n}\ell_1 - i = l + \ell_0 - 1 + \tilde{n}\ell_1 - i - \ell_0 + 1.
$$

Since $\ell_0 \geq 2\ell_1 + 2$ and $i \geq 0$ we have that

$$
\begin{aligned}
l + \tilde{n}\ell_1 - i &\leq l + \ell_0 - 1 + \tilde{n}\ell_1 - 2\ell_1 - 2 + 1 \leq l + \ell_0 - 1 + (\tilde{n} - 2)\ell_1 - 1 \\
&\leq l + \ell_0 - 1 + (\tilde{n} - 3)\ell_1 + (\ell_1 - 1)
\end{aligned}
$$

and by (37) and (38) with $n \leq \tilde{n} - 3$ we have that $H_{l+\tilde{n}\ell_1-i} = O(y^{l-\tilde{n}+1})$ which yields

$$
\frac{\partial H_{l+\tilde{n}\ell_1-i}}{\partial y} = O(y^{l-\tilde{n}}).
$$

Therefore, using also the induction hypothesis, we have that

$$
\begin{aligned}
&y\frac{\partial H_{l+\ell_0-1+\tilde{n}\ell_1}}{\partial x} = -\bar{f}_{\ell_1} x^{\ell_1+1} \frac{\partial H_{l+\ell_0-1+(\tilde{n}-1)\ell_1}}{\partial x} + O(y^{l-\tilde{n}}) \\
&= (-1)^{\tilde{n}+1} \bar{f}_{\ell_1}^{\tilde{n}} a_l g_{\ell_0} K_{l,\tilde{n}-1} (\ell_0 + 1 + (\tilde{n} - 1)(\ell_1 + 1)) x^{\ell_0+\tilde{n}(\ell_1+1)} y^{l-2-\tilde{n}+1} + O(y^{l-\tilde{n}}).
\end{aligned}
$$

Hence,

$$
H_{l+\ell_0-1+\tilde{n}\ell_1} = (-1)^{\tilde{n}+1} \bar{f}_{\ell_1}^{\tilde{n}} a_l g_{\ell_0} K_{l,\tilde{n}} x^{\ell_0+1+\tilde{n}(\ell_1+1)} y^{l-2-\tilde{n}} + O(y^{l-\tilde{n}-1}),
$$

where

$$
K_{l,\tilde{n}} = \frac{(\ell_0 + 1 + (\tilde{n} - 1)(\ell_1 + 1))K_{l,\tilde{n}-1}}{\ell_0 + 1 + \tilde{n}(\ell_1 + 1)}.
$$

This proves (37) with $n = \tilde{n}$.

Computing the degree $l+\ell_0-1+\tilde{n}\ell_1+j$ in (30) with $j=1,\dots,\ell_1-1$, and the induction hypothesis, we have

$$
\begin{aligned}
y\frac{\partial H_{l+\ell_0-1+\tilde{n}\ell_1+j}}{\partial x} = &-\sum_{i=0}^{\ell_0+j-1+(\tilde{n}-1)\ell_1} \bar{f}_{\ell_1+i}x^{\ell_1+1+i}\frac{\partial H_{l+\ell_0-1-i+j+(\tilde{n}-1)\ell_1}}{\partial x} \\
&-\sum_{i=0}^{\tilde{n}\ell_1+j} g_{\ell_0+i}x^{\ell_0+i}\frac{\partial H_{l+\tilde{n}\ell_1+j-i}}{\partial y}.
\end{aligned} \tag{40}
$$

Note that since $i\ge 1$, by (37) and (38) with $n\le \tilde{n}-1$ we have

$$
\frac{\partial H_{l+\ell_0-1-i+j+(\tilde{n}-1)\ell_1}}{\partial x} = O(y^{l-(\tilde{n}-1)-2}) = O(y^{l-\tilde{n}-1}).
$$

Furthermore we can write $l+\tilde{n}\ell_1+j-i$ in $\partial H_{l+\tilde{n}\ell_1+j-i}/\partial y$ of the form

$$
l+\tilde{n}\ell_1+j-i = l+\ell_0-1+\tilde{n}\ell_1+j-i-\ell_0+1.
$$

Since $\ell_0\ge 2\ell_1+2$ and $i\ge 0$ we have that

$$
l+\tilde{n}\ell_1+j-i \le l+\ell_0-1+j+\tilde{n}\ell_1-2\ell_1-2+1 \le l+\ell_0-1+(\tilde{n}-2)\ell_1+j-1
$$

and by (38) with $n\le\tilde{n}-2$ we have that $H_{l+\tilde{n}\ell_1+j-i}=O(y^{l-\tilde{n}})$ which yields

$$
\frac{\partial H_{l+\tilde{n}\ell_1+j-i}}{\partial y} = O(y^{l-\tilde{n}-1}).
$$

Now simplifying (40) by y and integrating in x we conclude that (38) holds for $n=\tilde{n}$ and $j=1,\dots,\ell_1-1$. This completes the proof of the lemma. □

Now we continue with the proof of Proposition 15. Given $l\ge 2$ if we take $n=l-2$ and use (34) (if $l=2$) or (37) in Lemma 16 if $l\ge 3$ we get that

$$
H_{l+\ell_0-1+(l-2)\ell_1} = (-1)^{l-1}\bar{f}_{\ell_1}^{l-2}a_l g_{\ell_0}K_{1,l-2}x^{\ell_0+1+(l-2)(\ell_1+1)} + O(y),
$$

where $K_{1,0}=C_l$. Now computing in (30) the term of degree $l+\ell_0-1+(l-1)\ell_1$ and proceeding as in the proof of Lemma 16 we get that

$$
\begin{aligned}
&y\frac{\partial H_{l+\ell_0-1+(l-1)\ell_1}}{\partial x} \\
&= (-1)^l\bar{f}_{\ell_1}^{l-1}a_l g_{\ell_0}K_{1,l-2}(\ell_0+1+(l-2)(\ell_1+1))x^{\ell_0+(l-2)(\ell_1+1)} + O(y).
\end{aligned} \tag{41}
$$

Now setting $y=0$ in (41) and since $g_{\ell_0}\bar{f}_{\ell 1}K_{l,l-2}(\ell_0+1+(l-2)(\ell_1+1))\neq 0$ we get that $a_l=0$. This concludes the proof of the proposition. □

Appendix. Poincaré in [10] defined the notion of a center for a real polynomial differential system in $\mathbb{R}^2$. He showed that a necessary and sufficient condition in order that a real polynomial differential system has a center at a singular point with purely imaginary eigenvalues, is that it has a local analytic first integral defined in its neighborhood. Then doing a linear change of variables and a scaling of the independent variable, we can write any polynomial differential system with a focus at the origin with purely imaginary eigenvalues (i.e. having either weak focus or a center) into the form

$$
\begin{aligned}
\dot{x} &= p(x,y) = y + p_2(x,y) + \ldots + p_m(x,y), \\
\dot{y} &= q(x,y) = -x + q_2(x,y) + \ldots + q_m(x,y),
\end{aligned} \tag{42}
$$

where

$$p_i(x,y) = \sum_{j=0}^{i} a_{ij} x^{i-j} y^j, \quad q_i(x,y) = \sum_{j=0}^{i} b_{ij} x^{i-j} y^j.$$

The following result is due to Shi [13, 14]. It is better stated in [12] (see also [7]).

Lemma 17. *For the polynomial system* (42) *there exists a formal power series* $F \in \mathbb{Q}[a_{20}, \dots, b_{0m}][[x,y]]$,

$$F = \frac{1}{2}(x^2+y^2) + F_3(x,y) + F_4(x,y) + \dots,$$

and polynomials $V_1, \dots, V_i, \dots \in Q[a_{20}, \dots, b_{0m}]$ *such that*

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} p + \frac{\partial F}{\partial y} q = \sum_{i=1}^{\infty} V_i (x^2+y^2)^{i+1}.$$

The constants V_i are called focus quantities or Poincaré-Liapunov constants. They are not uniquely determined. From a result of Shi, all such V_i's are in the same coset modulo the ideal generated by $V_1, \dots, V_{i-1}$ in the ring $Q[a_{20}, \dots, b_{0m}]$. From Poincaré [10] system (42) has a center at the origin if and only if $V_i = 0$ for all i. The ideal $\langle V_1, \dots, V_i, \dots \rangle$ has a finite basis due to Hilbert's basis theorem.

A hyperbolic saddle such that the trace of its linear part is zero is a weak saddle. Dulac [3] in his studies of hyperbolic saddles in complex systems used a definition which in real coefficients can be a center. With a linear change of coordinates and a scaling of the independent variable, we can write any polynomial differential system with a weak saddle at the origin into the form

$$\begin{aligned} \dot{x} &= p(x,y) = y + p_2(x,y) + \dots + p_m(x,y), \\ \dot{y} &= q(x,y) = x + q_2(x,y) + \dots + q_m(x,y), \end{aligned} \tag{43}$$

where p_i and q_i are the same as in (42).

The proof of the next result is analogous to the proof of Lemma 17.

Lemma 18. *For the polinomial system* (43) *there exists a formal power series* $F \in \mathbb{Q}[a_{20}, \dots, b_{0m}][[x,y]]$,

$$F = \frac{1}{2}xy + F_3(x,y) + F_4(x,y) + \dots,$$

and polynomials $L_1, \dots, L_i, \dots \in Q[a_{20}, \dots, b_{0m}]$ *such that*

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} p + \frac{\partial F}{\partial y} q = \sum_{i=1}^{\infty} L_i (xy)^{i+1}.$$

The constants L_i are called the saddle quantities. They are not uniquely determined. As for the V_i's, all L_i's are in the same coset modulo the ideal generated by $L_1, \dots, L_{i-1}$ in the ring $Q[a_{20}, \dots, b_{0m}]$.

We say that the origin is an integrable saddle if and only if $L_i = 0$ for all i. The ideal $\langle L_1, \dots, L_i, \dots \rangle$ has a finite basis due to Hilbert's basis theorem.

Doing a linear change of coordinates any complex polynomial differential system with a weak saddle at the origin can be written as

$$\begin{aligned} \dot{x} &= y + \overline{p}(x,y), \\ \dot{y} &= x + \overline{q}(x,y), \end{aligned} \tag{44}$$

where $\overline{p}$ and $\overline{q}$ are complex polynomials without constant and linear terms. With the change of variables $x = a$, $y = ib$, system (44) becomes the complex differential system

$$\begin{aligned} \dot{a} &= ib + P(a, ib), \\ \dot{b} &= -ia + Q(a, ib), \end{aligned} \tag{45}$$

where P and Q are complex polynomials. Then the focus quantities V_j of system (45) coincide with the saddle quantities L_j of system (44). From this duality between focus quantities and saddle quantities it follows that an integrable saddle has an analytic first integral defined in a neighborhood of it. This is the reason why we call such a saddle an integrable saddle. The complex change (45) is done for showing the duality of weak focus and weak saddles.

If we apply a change of the form (45), precisely the change (15), to system (14) we get the differential system (16). Exactly system (16), which is a particular case of system (45), has been studied in [2, 15] showing that it has a first integral of the form given in Lemma 17 if and only if $\tilde{F}_j = 0$ for j odd.

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Received December 2014; revised July 2015.

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