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# ON THE ANALYTIC INTEGRABILITY OF THE LIÉNARD ANALYTIC DIFFERENTIAL SYSTEMS

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Abstract. We consider the Liénard analytic differential systems $\dot{x} = y$, $\dot{y} = -g(x) - f(x)y$, where $f, g : \mathbb{R} \to \mathbb{R}$ are analytic functions and the origin is an isolated singular point. Then for such systems we characterize the existence of local analytic first integrals in a neighborhood of the origin and the existence of global analytic first integrals.

1. **Introduction and statement of the main results.** One of the more classical problems in the qualitative theory of planar analytic differential systems in $\mathbb{R}^2$ is to characterize the existence of analytic first integrals in a neighborhood of an isolated singular point, and in particular the existence of a global analytic first integral when the differential system is defined in the whole $\mathbb{R}^2$.

One of the best and oldest results in this direction is the analytic nondegerate center theorem. In order to be more precise we recall some definitions. A singular point is a *nondegenerate center* if it is a center with eigenvalues purely imaginary. If a real planar analytic system has a nondegenerate center at the origin, then after a linear change of variables and a rescaling of the time variable, it can be written in the form:

$$\begin{array}{rcl} \dot{x} & = & y + X(x, y), \\ \dot{y} & = & -x + Y(x, y), \end{array} \tag{1}$$

where $X(x, y)$ and $Y(x, y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin.

Let $U$ be an open subset of $\mathbb{R}^2$, $H : U \to \mathbb{R}$ be a nonconstant analytic function and $\mathcal{X}$ be an analytic vector field defined on $U$. Then $H$ is an *analytic first integral* of $\mathcal{X}$ if $H$ is constant on the solutions of $\mathcal{X}$; i.e. if $\mathcal{X}H = 0$.

The next result is due to Poincaré [9] and Liapunov [6], see also Moussu [8].

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