ESAIM: COCV 22 (2016) 872–882
DOI: 10.1051/cocv/2015035

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ON THE POLYNOMIAL INTEGRABILITY OF A SYSTEM MOTIVATED BY THE RIEMANN ELLIPSOID PROBLEM

Jaume Llibre[1] and Clàudia Valls[2]

Abstract. We consider differential systems obtained by coupling two Euler–Poinsot systems. The motivation to consider such systems can be traced back to the Riemann ellipsoid problem. We provide new cases for which these systems are completely integrable. We also prove that these systems either are completely integrable or have at most four functionally independent analytic first integrals.

Mathematics Subject Classification. 34C05, 34A34, 34C14.

Received October 13, 2014.

1. Introduction and statement of the main results

Consider the following system of differential equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \nabla_x G(x,y)\wedge x,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \nabla_y G(x,y)\wedge y,\end{aligned}\tag{1.1}$$

where $(x,y)\in\mathbb{R}^3\times\mathbb{R}^3$ and G is the quadratic form

$$G=\frac{1}{2}\sum_{i=1}^{3}(a_i(x_i^2+y_i^2)+2b_ix_iy_i).$$

The a_i, b_i, $i=1,2,3$ are real constants. To avoid the trivial cases, at least one of the coupling constants b_i's is assumed to be different from zero. Of course, $x=0$ (or $y=0$) is an invariant subspace and here system (1.1) reduces to the Euler–Poinsot equations, see for instance [1,2]. The motivation to consider such systems can be traced back to the Riemann ellipsoid problem, see [6,8], and for more details the last part of Section 1 of the paper [9] where one page is dedicated to explain this connection, and also the papers [3,10]. Other results on the integrability of quadratic Hamiltonian systems similar to system (1.1) can be found in [11].

Keywords and phrases. Polynomial first integrals, homogeneous differential systems, Riemann ellipsoid problem, Euler–Poinsot systems, complete integrability.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain. jllibre@mat.uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal. cvalls@math.ist.utl.pt