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Liouvillian first integrals for a class of generalized Liénard polynomial differential systems

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We study the existence of Liouvillian first integrals for the generalized Liénard polynomial differential systems of the form $x' = y$, $y' = -g(x) - f(x)y$, where $f(x) = 3Q(x)Q'(x)P(x) + Q(x)^2P'(x)$ and $g(x) = Q(x)Q'(x)(Q(x)^2P(x)^2 - 1)$ with $P, Q \in \mathbb{C}[x]$. This class of generalized Liénard polynomial differential systems has the invariant algebraic curve $(y + Q(x)P(x))^2 - Q(x)^2 = 0$ of hyperelliptic type.

Keywords: Darboux polynomial; invariant algebraic curve; exponential factor; Liouvillian first integral; Liénard polynomial differential system

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1. Introduction and statement of the main result

One of the most classical and difficult problems in the qualitative theory of planar differential systems depending on parameters is to characterize the existence and non-existence of first integrals in functions of the parameters of the system.

We consider the polynomial differential system

$$x' = y, \qquad y' = -g(x) - f(x)y, \tag{1.1}$$

called the *generalized Liénard polynomial differential system*, where x and y are complex variables and the prime denotes the derivative with respect to the time t, which can be real or complex. Such differential systems appear in several branches of the sciences, such as biology, chemistry, mechanics and electronics (see, for example, [8,21] and the references therein). For $g(x) = x$ the Liénard differential system (1.1) is called the *classical Liénard polynomial differential system*.

Let

$$X = y\frac{\partial}{\partial x} - (g(x) + f(x)y)\frac{\partial}{\partial y}$$

be the polynomial vector field associated with system (1.1). Let U be an open and dense set in $\mathbb{C}^2$. We say that the non-locally constant function $H\colon U \to \mathbb{C}$ is a *first*