# LIMIT CYCLES FOR A VARIANT OF A GENERALIZED RICCATI EQUATION 

JAUME LLIBRE ${ }^{1}$ AND CLAUDIA VALLS ${ }^{2}$


#### Abstract

In this paper we provide a lower bound for the maximum number of limit cycles surrounding the origin of systems ( $\dot{x}, \dot{y}=\ddot{x}$ ) given by a variant of the generalized Riccati equation $$
\ddot{x}+\varepsilon x^{2 n+1} \dot{x}+b x^{4 n+3}=0,
$$ where $b>0, b \in \mathbb{R}, n$ is a non-negative integer and $\varepsilon$ is a small parameter. The tool for proving this result uses Abelian integrals.


## 1. Introduction and statement of the main results

Some variants of the generalized Riccati equation

$$
\begin{equation*}
\ddot{x}+\alpha x^{2 n+1} \dot{x}+x^{4 n+3}=0, \tag{1}
\end{equation*}
$$

have been studied for several authors, see for instance [8], [5], and the references quoted there. In the first paper the authors studied mainly the following variant of equation (1)

$$
\ddot{x}+(2 n+3) x^{2 n+1} \dot{x}+x^{4 n+3}+\omega^{2} x=0,
$$

showing numerically that such differential equation exhibits isochronous oscillations. In the second paper the authors study the variant of equation (1)

$$
\ddot{x}+(2 n+3) x^{2 n+1} \dot{x}+x^{4 n+3}+\omega^{2} x^{2 n+1}=0,
$$

and they find the analytical expression of some particular solutions.
In the present paper we will study the following variant of the generalized Riccati equation (1)

$$
\begin{equation*}
\ddot{x}+\varepsilon x^{2 n+1} \dot{x}+b x^{4 n+3}+\varepsilon a(x+y q(x))=0, \tag{2}
\end{equation*}
$$

where $a, b \in \mathbb{R}$ with $b \neq 0, n$ is a non-negative integer, $\varepsilon$ is a small parameter and $q(x)$ is a polynomial of degree $m$.

[^0]Equation (1) can be written as

$$
\begin{align*}
& \dot{x}=y, \\
& \dot{y}=-b x^{4 n+3}-\varepsilon a(x+y q(x))-\varepsilon x^{2 n+1} y, \tag{3}
\end{align*}
$$

or equivalently in the form

$$
\begin{align*}
\dot{x} & =-\frac{\partial H}{\partial y}+\varepsilon P(x, y)  \tag{4}\\
\dot{y} & =\frac{\partial H}{\partial x}+\varepsilon Q(x, y)
\end{align*}
$$

where

$$
\begin{equation*}
H(x, y)=\frac{y^{2}}{2}+\frac{b x^{4 n+4}}{4 n+4} \tag{5}
\end{equation*}
$$

and

$$
P(x, y)=0, \quad Q(x, y)=-a(x+y q(x))-x^{2 n+1} y
$$

Observe that for $b>0$ there is a family of ovals $\gamma_{h} \subset H^{-1}(h)$ continuously depending on a parameter $h>0$ and varying in the compact components of $H^{-1}(h)$. Moreover all the ovals $\gamma_{h}$ fill up the plane $\mathbb{R}^{2}$ when $h$ varies on all positive real numbers. These ovals are periodic orbits of the Hamiltonian system (4) with $\varepsilon=0$.

The objective of this paper is to find the maximum number of values of $h$ (that we denote by $h^{*}$ ) for which it bifurcate from $\gamma_{h^{*}}$ a limit cycle of the differential system (4) for $|\varepsilon|>0$ sufficiently small.

Theorem 1. For $|\varepsilon|>0$ sufficiently small there are systems (3) with $b>0$ having $m$ limit cycles $\Gamma_{h_{m}^{*}}$ that when $\varepsilon \rightarrow 0$ tend to periodic orbits $\gamma_{h_{m}^{*}}$ of the Hamiltonian system (3) with $b>0$ and $\varepsilon=0$. Moreover there are polynomials $q(x)$ for which the differential system (3) with $b>0$ and $|\varepsilon|>0$ sufficiently small has exactly $m$ limit cycles.

The proof of Theorem 1 is given in section 2 .
Note that Theorem 1 is closely related to the weakened 16th Hilbert problem proposed by Arnold in $[1,2]$ which in its turn is closely related to determining an upper bound for the number of limit cycles of a perturbed Hamiltonian system of the form in (4). For other papers on limit cycles see for instance $[6,7]$ and the references quoted there.

## 2. Proof of Theorem 1

To prove Theorem 1 we will use use the following Theorem whose proof can be obtained, for instance, in [4].

Theorem 2. Assume that there is a family of ovals $\gamma_{h} \subset H^{-1}(h)$, continuously depending on $h \in(a, b)$. Define the Abelian integral as

$$
\begin{equation*}
I(h)=\int_{\gamma_{h}} P(x, y) d y-Q(x, y) d x . \tag{6}
\end{equation*}
$$

If there exists an $h^{*} \in(a, b)$ such that $I\left(h^{*}\right)=0$ and $I^{\prime}\left(h^{*}\right) \neq 0$ then for $\varepsilon$ sufficiently small, the Hamiltonian system (4) has at most one limit cycle $\Gamma_{h^{*}}$ which tends to $\gamma_{h^{*}}$ as $\varepsilon \rightarrow 0$.

We first write the polynomial $q(x)=\sum_{i=0}^{n} q_{j} x^{j}$. Note that the unperturbed system (2) (with $\varepsilon=0$ ) is Hamiltonian with the Hamiltonian $H$ given in (5). The periodic orbits of the unperturbed system (2) with $h>0$ are the ovals $\gamma_{h}$. Now we will use Theorem 2 and so we shall compute the Abelian integral $I(h)$ given in (6). We have

$$
\begin{aligned}
I(h)= & \iint_{H(x, y) \leq h} \frac{\partial}{\partial y}\left(x^{2 n+1} y+a(x+y q(x))\right) d x d y \\
= & \iint_{H(x, y) \leq h}\left(x^{2 n+1}+a q(x)\right) d x d y \\
= & \iint_{H(x, y) \leq h} x^{2 n+1} d x d y+a \sum_{i=0}^{m} q_{i} \iint_{H(x, y) \leq h} x^{i} d x d y \\
= & 2 \int_{-\bar{x}}^{\bar{x}} x^{2 n+1}\left(2 h-\frac{b}{2(n+1)} x^{4(n+1)}\right)^{1 / 2} d x \\
& +2 a \sum_{i=0}^{m} q_{i} \int_{-\bar{x}}^{\bar{x}} x^{i}\left(2 h-\frac{b}{2(n+1)} x^{4(n+1)}\right)^{1 / 2} d x,
\end{aligned}
$$

where

$$
\bar{x}=\left(\frac{4 h(n+1)}{b}\right)^{1 /(4(n+1))} .
$$

Note that for any integer $j$ we have

$$
\begin{aligned}
& \int_{-\bar{x}}^{\bar{x}} x^{j}\left(2 h-\frac{b}{2(n+1)} x^{4(n+1)}\right)^{1 / 2} d x= \\
& \quad \frac{2^{\frac{j-n}{2 n+2}}\left(1+(-1)^{j}\right) b^{-\frac{j+1}{4 n+4}}(n+1)^{\frac{j+1}{4 n+4}} \sqrt{\pi} \Gamma\left(\frac{j+4 n+5}{4 n+4}\right)}{(j+1) \Gamma\left(\frac{j+6 n+7}{4 n+4}\right)} h^{\frac{j+4 n+5}{4 n+4}},
\end{aligned}
$$

being $\Gamma(\cdot)$ the Gamma function. If $\bar{h}=h^{1 /(4(n+1))}$ and

$$
B_{j, n}=\frac{2^{\frac{j-n}{2 n+2}}\left(1+(-1)^{j}\right) b^{-\frac{j+1}{4 n+4}}(n+1)^{\frac{j+1}{4 n+4}} \sqrt{\pi} \Gamma\left(\frac{j+4 n+5}{4 n+4}\right)}{(j+1) \Gamma\left(\frac{j+6 n+7}{4 n+4}\right)}
$$

Then

$$
J(\bar{h})=I(h)=2 \bar{h}^{4 n+5}\left(B_{2 n+1, n} \bar{h}^{2 n+1}+a \sum_{i=0}^{m} q_{i} B_{i, n} \bar{h}^{i}\right) .
$$

Note that $J(\bar{h})$ has at most $m$ simple positive zeros if $m \geq 2 n+1$, and by generalized Descartes theorem (see the Appendix) $J(\bar{h})$ has at most $m$ simple positive zeros if $m<2 n+1$.

Since the coefficients $q_{i}$ are arbitrary, we can choose a perturbation $q(x)$ in such a way that $J(\bar{h})$ has exactly $m$ simple positive zeros, and consequently there are differential systems (3) with $m$ limit cycles. This concludes the proof of the theorem.

## 3. Appendix

We recall the Descartes Theorem about the number of zeros of a real polynomial (for a proof see for instance [3]).

Descartes Theorem Consider the real polynomial $p(x)=a_{i_{1}} x^{i_{1}}+$ $a_{i_{2}} x^{i_{2}}+\cdots+a_{i_{r}} x^{i_{r}}$ with $0 \leq i_{1}<i_{2}<\cdots<i_{r}$ and $a_{i_{j}} \neq 0$ real constants for $j \in\{1,2, \cdots, r\}$. When $a_{i_{j}} a_{i_{j+1}}<0$, we say that $a_{i_{j}}$ and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is $r-1$, then $p(x)$ has at most $m$ positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly r-1 positive real roots.

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${ }^{1}$ Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain.

E-mail address: jllibre@mat.uab.cat
2 Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Av. Rovisco Pais 1049-001, Lisboa, PortUGAL

E-mail address: cvalls@math.ist.utl.pt


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