LIMIT CYCLES FOR A VARIANT OF A GENERALIZED RICCATI EQUATION

JAUME LLIBRE¹ AND CLAUDIA VALLS²

ABSTRACT. In this paper we provide a lower bound for the maximum number of limit cycles surrounding the origin of systems $(\dot{x}, \dot{y} = \ddot{x})$ given by a variant of the generalized Riccati equation

$$\ddot{x} + \varepsilon x^{2n+1} \dot{x} + b x^{4n+3} = 0,$$

where b > 0, $b \in \mathbb{R}$, *n* is a non–negative integer and ε is a small parameter. The tool for proving this result uses Abelian integrals.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Some variants of the generalized Riccati equation

(1)
$$\ddot{x} + \alpha x^{2n+1} \dot{x} + x^{4n+3} = 0,$$

have been studied for several authors, see for instance [8], [5], and the references quoted there. In the first paper the authors studied mainly the following variant of equation (1)

$$\ddot{x} + (2n+3)x^{2n+1}\dot{x} + x^{4n+3} + \omega^2 x = 0,$$

showing numerically that such differential equation exhibits isochronous oscillations. In the second paper the authors study the variant of equation (1)

$$\ddot{x} + (2n+3)x^{2n+1}\dot{x} + x^{4n+3} + \omega^2 x^{2n+1} = 0,$$

and they find the analytical expression of some particular solutions.

In the present paper we will study the following variant of the generalized Riccati equation (1)

(2)
$$\ddot{x} + \varepsilon x^{2n+1} \dot{x} + bx^{4n+3} + \varepsilon a(x+yq(x)) = 0,$$

where $a, b \in \mathbb{R}$ with $b \neq 0$, *n* is a non–negative integer, ε is a small parameter and q(x) is a polynomial of degree *m*.

Key words and phrases. generalized Riccati system, Limit cycles, weak 16th Hilbert problem, Abelian integral.





²⁰¹⁰ Mathematics Subject Classification. 37G15, 37D45.

Equation (1) can be written as

(3)
$$\dot{x} = y,$$

$$\dot{y} = -bx^{4n+3} - \varepsilon a(x + yq(x)) - \varepsilon x^{2n+1}y,$$

or equivalently in the form

(4)
$$\dot{x} = -\frac{\partial H}{\partial y} + \varepsilon P(x, y),$$
$$\dot{y} = \frac{\partial H}{\partial x} + \varepsilon Q(x, y),$$

where

(5)
$$H(x,y) = \frac{y^2}{2} + \frac{bx^{4n+4}}{4n+4},$$

and

$$P(x,y) = 0, \quad Q(x,y) = -a(x+yq(x)) - x^{2n+1}y.$$

Observe that for b > 0 there is a family of ovals $\gamma_h \subset H^{-1}(h)$ continuously depending on a parameter h > 0 and varying in the compact components of $H^{-1}(h)$. Moreover all the ovals γ_h fill up the plane \mathbb{R}^2 when h varies on all positive real numbers. These ovals are periodic orbits of the Hamiltonian system (4) with $\varepsilon = 0$.

The objective of this paper is to find the maximum number of values of h (that we denote by h^*) for which it bifurcate from γ_{h^*} a limit cycle of the differential system (4) for $|\varepsilon| > 0$ sufficiently small.

Theorem 1. For $|\varepsilon| > 0$ sufficiently small there are systems (3) with b > 0 having m limit cycles $\Gamma_{h_m^*}$ that when $\varepsilon \to 0$ tend to periodic orbits $\gamma_{h_m^*}$ of the Hamiltonian system (3) with b > 0 and $\varepsilon = 0$. Moreover there are polynomials q(x) for which the differential system (3) with b > 0 and $|\varepsilon| > 0$ sufficiently small has exactly m limit cycles.

The proof of Theorem 1 is given in section 2.

Note that Theorem 1 is closely related to the weakened 16th Hilbert problem proposed by Arnold in [1, 2] which in its turn is closely related to determining an upper bound for the number of limit cycles of a perturbed Hamiltonian system of the form in (4). For other papers on limit cycles see for instance [6, 7] and the references quoted there.

2. Proof of Theorem 1

To prove Theorem 1 we will use use the following Theorem whose proof can be obtained, for instance, in [4].

 $\mathbf{2}$

Theorem 2. Assume that there is a family of ovals $\gamma_h \subset H^{-1}(h)$, continuously depending on $h \in (a, b)$. Define the Abelian integral as

(6)
$$I(h) = \int_{\gamma_h} P(x, y) \, dy - Q(x, y) \, dx.$$

If there exists an $h^* \in (a, b)$ such that $I(h^*) = 0$ and $I'(h^*) \neq 0$ then for ε sufficiently small, the Hamiltonian system (4) has at most one limit cycle Γ_{h^*} which tends to γ_{h^*} as $\varepsilon \to 0$.

We first write the polynomial $q(x) = \sum_{i=0}^{n} q_j x^j$. Note that the unperturbed system (2) (with $\varepsilon = 0$) is Hamiltonian with the Hamiltonian H given in (5). The periodic orbits of the unperturbed system (2) with h > 0 are the ovals γ_h . Now we will use Theorem 2 and so we shall compute the Abelian integral I(h) given in (6). We have

$$\begin{split} I(h) &= \int \int_{H(x,y) \le h} \frac{\partial}{\partial y} (x^{2n+1}y + a(x + yq(x))) \, dx \, dy \\ &= \int \int_{H(x,y) \le h} (x^{2n+1} + aq(x)) \, dx \, dy \\ &= \int \int_{H(x,y) \le h} x^{2n+1} \, dx \, dy + a \sum_{i=0}^{m} q_i \int \int_{H(x,y) \le h} x^i \, dx \, dy \\ &= 2 \int_{-\overline{x}}^{\overline{x}} x^{2n+1} \left(2h - \frac{b}{2(n+1)} x^{4(n+1)} \right)^{1/2} \, dx \\ &+ 2a \sum_{i=0}^{m} q_i \int_{-\overline{x}}^{\overline{x}} x^i \left(2h - \frac{b}{2(n+1)} x^{4(n+1)} \right)^{1/2} \, dx, \end{split}$$

where

$$\overline{x} = \left(\frac{4h(n+1)}{b}\right)^{1/(4(n+1))}$$

Note that for any integer j we have

$$\int_{-\overline{x}}^{\overline{x}} x^{j} \left(2h - \frac{b}{2(n+1)} x^{4(n+1)}\right)^{1/2} dx = \frac{2^{\frac{j-n}{2n+2}} \left(1 + (-1)^{j}\right) b^{-\frac{j+1}{4n+4}} (n+1)^{\frac{j+1}{4n+4}} \sqrt{\pi} \Gamma\left(\frac{j+4n+5}{4n+4}\right)}{(j+1)\Gamma\left(\frac{j+6n+7}{4n+4}\right)} h^{\frac{j+4n+5}{4n+4}},$$

being $\Gamma(\cdot)$ the Gamma function. If $\overline{h} = h^{1/(4(n+1))}$ and

$$B_{j,n} = \frac{2^{\frac{j-n}{2n+2}} \left(1 + (-1)^{j}\right) b^{-\frac{j+1}{4n+4}} (n+1)^{\frac{j+1}{4n+4}} \sqrt{\pi} \Gamma\left(\frac{j+4n+5}{4n+4}\right)}{(j+1)\Gamma\left(\frac{j+6n+7}{4n+4}\right)}$$

Then

$$J(\overline{h}) = I(h) = 2\overline{h}^{4n+5} \left(B_{2n+1,n}\overline{h}^{2n+1} + a \sum_{i=0}^{m} q_i B_{i,n}\overline{h}^i \right).$$

Note that $J(\overline{h})$ has at most m simple positive zeros if $m \ge 2n+1$, and by generalized Descartes theorem (see the Appendix) $J(\overline{h})$ has at most m simple positive zeros if m < 2n + 1.

Since the coefficients q_i are arbitrary, we can choose a perturbation q(x) in such a way that $J(\overline{h})$ has exactly m simple positive zeros, and consequently there are differential systems (3) with m limit cycles. This concludes the proof of the theorem.

3. Appendix

We recall the Descartes Theorem about the number of zeros of a real polynomial (for a proof see for instance [3]).

Descartes Theorem Consider the real polynomial $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ with $0 \le i_1 < i_2 < \cdots < i_r$ and $a_{i_j} \ne 0$ real constants for $j \in \{1, 2, \cdots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is r-1, then p(x) has at most m positive real roots. Moreover, it is always possible to choose the coefficients of p(x) in such a way that p(x) has exactly r-1 positive real roots.

Acknowledgements

The first author is partially supported by a MINECO grant number MTM2013-40998-P, an AGAUR grant number 2014SGR-568 and the grants FP7-PEOPLE-2012-IRSES 318999 and 316338. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2013.

References

- V.I. ARNOLD, Loss of stability of self-oscillations close to resonance and versal deformations of equivariant vector fields Funct. Anal. Appl. 11 (1977), 85–92.
- [2] V.I. ARNOLD, Ten problems, Adv. Soviet. Math. 1 (1990), 1–8.
- [3] I.S. BEREZIN AND N.P. ZHIDKOV, Computing Methods, Volume II, Pergamon Press, Oxford, 1964.
- [4] C. CHRISTOPHER AND C. LI, *Limit cycles of differential equations*, Advances Courses in Mathematics, CRM, Barcelona, Birkhäuser, Basel, 2002.

4

LIMIT CYCLES

- [5] A. GHOSE-CHOUDHURY AND P. GUHA, An analytic technique for the solutions of nonlinear oscillators with damping using the Abel Equation, arXiv:1608.02324v1.
- [6] J. LLIBRE AND L. ROBERTO, On the periodic orbits of the third-order differential equation $x''' - \mu x'' + x' - \mu x = \varepsilon F(x, x', x'')$, Applied Mathematics Letters **26** (2013),425–430.
- [7] J. LLIBRE AND M.A. TEIXEIRA, *Limit cycles bifurcating from a 2-dimensional isochronous cylinder*, Applied Mathematics Letters **22** (2009), 1231–1234.
- [8] A. SARKAR, P. GUHA, A. GHOSE-CHOUDHURY, J.K. BHATTACHARJEE, A.K. MALLIK AND P.G.L. LEACH, On the properties of a variant of the Riccati system of equations, J. Phys. A: Math. Theor. 45 (2012), 415101 (9 pp).

¹ Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain.

E-mail address: jllibre@mat.uab.cat

² DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, UNI-VERSIDADE TÉCNICA DE LISBOA, AV. ROVISCO PAIS 1049-001, LISBOA, POR-TUGAL

E-mail address: cvalls@math.ist.utl.pt