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ON THE INTEGRABILITY OF THE 5-DIMENSIONAL LORENZ SYSTEM FOR THE GRAVITY-WAVE ACTIVITY

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ABSTRACT. We consider the 5-dimensional Lorenz system

$$\begin{aligned} U' &= -VW + bVZ,\\ V' &= UW - bUZ,\\ W' &= -UV,\\ X' &= -Z,\\ Z' &= bUV + X, \end{aligned}$$

where $b \in \mathbb{R}\setminus\{0\}$ and the derivative is with respect to T. This system describes coupled Rosby waves and gravity waves. First we prove that the number of functionally independent global analytic first integrals of this differential system is two. This solves an open question in the paper, *On the analytic integrability of the* 5*-dimensional Lorenz system for the gravity-wave activity*, Proc. Amer. Math. Soc. **142** (2014), 531–537, where it was proved that this number was two or three. Moreover, we characterize all the invariant algebraic surfaces of the system, and additionally we show that it has only two functionally independent Darboux first integrals.

1. Introduction and statement of the main results

E. N. Lorenz constructed in [11] a 5-dimensional 1-parameter differential system in $\mathbb{R}^5$ which describes coupled Rosby waves and gravity waves:

$$\begin{aligned} U' &= -VW + bVZ,\\ V' &= UW - bUZ,\\ W' &= -UV,\\ X' &= -Z,\\ Z' &= bUV + X. \end{aligned} \tag{1}$$

He studied its slow manifolds and in this paper we are interested in studying its global analytic integrability, its algebraic invariant surfaces and its Darboux first

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