COMPUTING POLYNOMIAL SOLUTIONS OF EQUIVARIANT POLYNOMIAL ABEL DIFFERENTIAL EQUATIONS

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Abstract. Let $a(x)$ non-constant and $b_j(x)$ for $j = 0, 1, 2, 3$ be real or complex polynomials in the variable x. Then the real or complex equivariant polynomial Abel differential equations $a(x)\dot{y} = b_1(x)y + b_3(x)y^3$ with $b_3(x) \neq 0$, and the real or complex polynomial equivariant polynomial Abel differential equations of second kind $a(x)y\dot{y} = b_0(x) + b_2(x)y^2$ with $b_2(x) \neq 0$, have at most 7 polynomial solutions. Moreover there are equations of these type having these maximum number of polynomial solutions.

1. Introduction and statement of the main results

Abel differential equations of first kind

$$a(x)\dot{y} = b_0(x) + b_1(x)y + b_2(x)y^2 + b_3(x)y^3 \tag{1}$$

with $b_3(x) \neq 0$ appear in many text-books of ordinary differential equations as one of first non-trivial examples of nonlinear differential equations, see for instance [10]. Here the dot denotes the derivative with respect to the independent variable x. If $b_3(x) = b_0(x) = 0$ or $b_2(x) = b_0(x) = 0$ the Abel differential equation reduces to a Bernoulli differential equation, while if $b_3(x) = 0$ the Abel differential equation reduces to a Riccati differential equation.

The Abel differential equations (1) have been studied intensively, either calculating their solutions (see for instance [7, 11, 12, 13]), or classifying their centers (see [2, 3, 4]), and recently in [6, 8, 9] the authors studied the polynomial solutions of the differential equation $y' = \sum_{i=0}^{n} a_i(x)y^i$.

The analysis of particular solutions (as polynomial or rational solutions) of the differential equations is important for understanding the set of solutions of a differential equation. In 1936 Rainville [14] characterized the Riccati differential equations $\dot{y} = b_0(x) + b_1(x)y + y^2$, with $b_0(x)$ and $b_1(x)$ polynomials in the variable x, having polynomial solutions.

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Campbell and Golomb [5] in 1954 provides an algorithm for determining the polynomial solutions of the Riccati differential equation $a(x)y' = b_0(x)+b_1(x)y+b_2(x)y^2$, where a, b_0, b_1, b_2 are polynomials in the variable x. Behloul and Cheng [1] in 2006 gave a different algorithm for finding the rational solutions of the differential equations $a(x)y' = \sum_{i=0}^{n} b_i(x)y^i$, where a, b_i are polynomials in the variable x.

Here we consider the Abel differential equations (1) where $a(x) \in \mathbb{F}[x] \setminus \{0\}$, $b_i(x) \in \mathbb{F}[x]$, $i = 0, 1, 2, 3$, $b_3(x) \neq 0$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{F}[x]$ is the ring of polynomials in the variable x with coefficients in $\mathbb{F}$. We also assume that $a(x)$ is not constant. The case $a(x)$ constant has been studied in [9]. We say that the Abel differential equation (1) has degree η.

Equation (1) is *reversible* with respect to the change of variables $(x, y) \to (x, -y)$ if the following equation

$$-a(x)\dot{y} = -(b_0(x) - b_1(x)y + b_2(x)y^2 - b_3(x)y^3)$$

coincides with equation (1). In particular this implies $b_1(x) = b_3(x) = 0$, and since $b_3(x) = 0$ we do not consider these reversible differential equations.

The Abel differential equation (1) is *equivariant* with respect to the change of variables $(x, y) \to (x, -y)$ if the following equation

$$-a(x)\dot{y} = b_0(x) - b_1(x)y + b_2(x)y^2 - b_3(x)y^3$$

coincides with equation (1). This implies $b_0(x) = b_2(x) = 0$. In this paper first we focus our study in these kind of *equivariant polynomial Abel equations*, i.e. in the equations

$$a(x)\dot{y} = b_1(x)y + b_3(x)y^3. \tag{2}$$

Theorem 1. *Real or complex equivariant polynomial Abel differential equations with $b_3(x) \neq 0$ and $a(x)$ non-constant, have at most 7 polynomial solutions. Moreover there are equations of this type having these maximum number of polynomial solutions.*

The proof of Theorem 1 is given in section 2.

Our second objective in this paper is on the Abel differential equations of second kind, i.e. on the equations of the form

$$a(x)y\dot{y} = b_0(x) + b_1(x)y + b_2(x)y^2, \tag{3}$$

where $a(x), b_i(x) \in \mathbb{F}[x]$ for $i = 0, 1, 2$, with $a(x)$ and $b_2(x)$ non-zero. We also consider the ones that are equivariant with respect to the change $(x, y) \to (x, -y)$. Then we have that $b_1(x) = 0$ and so equation (3) becomes

$$a(x)y\dot{y} = b_0(x) + b_2(x)y^2. \tag{4}$$

We also assume that $a(x)$ is not constant, because the case $a(x)$ constant has been studied in [6]. We say that the *equivariant polynomial Abel differential equation of second kind* (4).

Theorem 2. *Real or complex equivariant polynomial Abel differential equations of second kind with $b_2(x) \neq 0$ and $a(x)$ non-constant, have at most 7 polynomial solutions. Moreover there are equations of this type having these maximum number of polynomial solutions.*

The proof of Theorem 2 is given in section 3.

2. Proof of Theorem 1

First we recall that if $y(x) \neq 0$ is a solution of equation (2), then $-y(x)$ is also a solution of equation (2) which is different from $y(x)$.

Lemma 3. *Let $y_0(x) \neq 0$, $y_1(x), y_2(x)$ be polynomial solutions of equation (2) such that $y_1(x) \not\equiv 0$, $y_2(x) \not\equiv 0$ and $y_2(x) \neq -y_1(x)$. Set $y_1(x) = g(x)\tilde{y}_1(x)$ and $y_2(x) = g(x)\tilde{y}_2(x)$ where $g = gcd(y_1, y_2)$. Then, except the solution $y = 0$, all the other polynomial solutions of equation (2) can be expressed as*

$$y_0(x; c) = \pm \frac{\tilde{y}_1(x)\tilde{y}_2(x)g(x)}{\big(c\tilde{y}_1^2(x) + (1-c)\tilde{y}_2^2(x)\big)^{1/2}}, \tag{5}$$

where c is a constant and $\big(c\tilde{y}_1^2(x) + (1-c)\tilde{y}_2^2(x)\big)^{1/2}$ is a polynomial.

Proof. Let y be a nonzero polynomial solution of equation (2). The functions $z_0 = 1/y_0^2$, $z_1 = 1/y_1^2$ and $z_2 = 1/y_2^2$ are solutions of a linear differential equation and satisfy

$$-a(x)\dot{z}_i = 2b_1(x)z_i + 2b_3(x), \quad i = 0, 1, 2.$$

Therefore we have

$$\frac{\dot{z}_0(x) - \dot{z}_1(x)}{z_0(x) - z_1(x)} = \frac{\dot{z}_2(x) - \dot{z}_1(x)}{z_2(x) - z_1(x)}.$$

Integrating this equality we obtain

$$z_0(x) = z_1(x) + c(z_2(x) - z_1(x)),$$

with c an arbitrary constant. So the general solution of equation (2) is

$$\begin{aligned} y_0^2(x) &= \frac{1}{z_0(x)} = \frac{1}{z_1(x) + c(z_2(x) - z_1(x))} \\ &= \frac{y_1^2(x)y_2^2(x)}{cy_1^2(x) + (1-c)y_2^2(x)} = \frac{\tilde{y}_1^2(x)\tilde{y}_2^2(x)g(x)^2}{c\tilde{y}_1^2(x) + (1-c)\tilde{y}_2^2(x)}, \end{aligned}$$

with c an arbitrary constant. □

In view of Lemma 3, if $y_1(x), y_2(x)$ are polynomial solutions of equation (2) such that $y_1(x) \not\equiv 0$, $y_2(x) \not\equiv 0$, $y_2(x) \neq -y_1(x)$, then any other polynomial solution different from them is of the form given in (5) for some appropriate constant c such that $c \notin \{0, 1\}$. In particular, $c\tilde{y}_1^2(x) + (1-c)\tilde{y}_2^2(x)$ is a square of a polynomial P and P divides g. We claim that this c is unique.

We write the condition that $c\tilde{y}_1^2(x)+(1-c)\tilde{y}_2^2(x)$ is a square of a polynomial in the form

$$\tilde{y}_1^2 + d\tilde{y}_2^2 = Z_1^2$$

where $d = (1-c)/c$ (we recall that $c \notin \{0,1\}$). We claim that there is a unique d for which this is possible.

For proving the claim we proceed by contradiction. Assume that there exists d_1, d_2 for which

$$\tilde{y}_1^2 + d_1\tilde{y}_2^2 = Z_1^2 \quad \text{and} \quad \tilde{y}_1^2 + d_2\tilde{y}_2^2 = Z_2^2, \tag{6}$$

for some polynomials Z_1, Z_2 and $d_1, d_2 \neq 1$ with $d_1 \neq d_2$. First we state and prove an auxiliary result.

Lemma 4. *The polynomial solutions of $X^2 + Y^2 = Z^2$ with pairwise co-prime polynomials X, Y, Z are of the form*

$$\pm X = 2ab, \quad \pm Y = a^2 - b^2, \quad \pm Z = a^2 + b^2$$

(or with X, Y interchanged) where a and b are co-prime polynomials.

Proof. It is sufficient to consider the plus case because if X, Y, Z is a solution so are $\pm X, \pm Y, \pm Z$. We can assume that are pairwise co-prime as, if a polynomial divides two of them it must divide the third and can be canceled from the identity. Let $X = 2u$, $Y + Z = 2v$ and $Y - Z = 2w$ with u, v, w polynomials and where v and w are coprime. It follows from the relation

$$X^2 = Z^2 - Y^2 = (Z+Y)(Z-Y)$$

that

$$u^2 = vw.$$

So v, w must be squares as they are co-prime. Let $v = a^2$ and $w = b^2$ where a and b are coprime. Hence

$$Z = a^2 + b^2, \quad Y = a^2 - b^2 \quad X = 2ab$$

and the lemma is proved. □

In the first identity in (6) using Lemma 4 we can write

$$\tilde{y}_1 = 2ab, \quad \sqrt{d_1}\tilde{y}_2 = a^2 - b^2,$$

where a and b are coprime. Then we can write the second identity in (6) as

$$\begin{aligned}
\tilde{y}_1^2 + d_2\tilde{y}_2^2 = \tilde{y}_1^2 + \Big(\frac{\sqrt{d_2}}{\sqrt{d_1}}(\sqrt{d_1}\tilde{y}_2)\Big)^2 &= 4a^2b^2 + \frac{d_2}{d_1}(a^2-b^2)^2 \\
&= \Big(\frac{d_2}{d_1} - 1\Big)(a^2-b^2) + (a^2+b^2) \\
&= \Big(\sqrt{(\frac{d_2}{d_1} - 1)}(a^2-b^2)\Big)^2 + (a^2+b^2)^2.
\end{aligned}$$

Since $d_2 \neq d_1$ we have that setting $\gamma = \sqrt{(\frac{d_2}{d_1} - 1)}$ then $\gamma \neq 0$. Let

$$a_1 = \sqrt{\gamma} a, \quad b_1 = \sqrt{\gamma} b.$$

Then

$\tilde{y}_1^2 + d_2 \tilde{y}_2^2 = (\gamma(a^2 - b^2))^2 + (a^2 + b^2)^2 = (a_1^2 - b_1^2)^2 + \gamma^{-2}(a_1^2 + b_1^2)^2 = Y_1^2 + X_1^2$.

In view of Lemma 4 since $Y_1 = a_1^2 - b_1^2$ we must have that $X_1 = 2a_1 b_1$. Therefore

$$X_1 = \gamma^{-1}(a_1^2 + b_1^2) = 2a_1 b_1, \quad \text{that is} \quad a_1^2 + b_1^2 - 2\gamma a_1 b_1 = 0.$$

This yields

$$a_1 = \gamma b_1 \pm b_1 \sqrt{\gamma^2 - 1} = b_1(\gamma \pm \sqrt{\gamma^2 - 1}).$$

Since $\gamma \pm \sqrt{\gamma^2 - 1} \neq 0$ and a and b are coprime (and so are a_1 and b_1) we get a contradiction. This proves the claim.

In short, there is at most one constant $c \notin \{0, 1\}$ such that $c\tilde{y}_1^2 + (1 - c)\tilde{y}_2^2$ is a square of a polynomial meaning that equation (2) has at most seven different polynomial solutions 0, $\pm y_1$, $\pm y_2$ and y_0 as in (5).

Example 1. We consider the equivariant polynomial Abel differential equation (2) with

$$\begin{aligned} a(x) =& \quad -2x + 48x^3 - 768x^7 + 512x^9, \\ b_1(x) =& \quad 2(-1 + 96x^4 - 1536x^6 + 768x^8), \\ b_3(x) =& \quad 64. \end{aligned}$$

This equation has the following 7 polynomial solutions

$$\begin{aligned} y_1(x) =& \quad 0, \\ y_{2,3}(x) =& \quad \pm \left(2\sqrt{2}x^3 + \frac{x}{\sqrt{2}} \right), \\ y_{4,5}(x) =& \quad \pm \left(x - 4x^3 \right), \\ y_{6,7}(x) =& \quad \pm \left(\frac{1}{4\sqrt{2}} - 2\sqrt{2}x^4 \right). \end{aligned}$$

3. Proof of Theorem 2

First we recall that if $y(x) \neq 0$ is a solution of equation (4), then $-y(x)$ is also a solution of equation (4) which is different from $y(x)$.

Lemma 5. *Let $y_0(x) \neq 0$, $y_1(x), y_2(x)$ be polynomial solutions of equation (4) such that $y_1(x) \not\equiv 0$, $y_2(x) \not\equiv 0$ and $y_2(x) \neq -y_1(x)$. Set $y_1(x) = g(x)\tilde{y}_1(x)$ and $y_2(x) = g(x)\tilde{y}_2(x)$ where $g = gcd(y_1, y_2)$. Then, except the solution $y = 0$, all the other polynomial solutions of equation (4) can be expressed as*

$$y_0(x; c) = \pm g(x)\big(c\tilde{y}_1^2(x) + (1 - c)\tilde{y}_2^2(x)\big)^{1/2}, \tag{7}$$

where c is a constant.

Proof. Let y be a nonzero polynomial solution of equation (2). The functions $z_0 = y_0^2$, $z_1 = y_1^2$ and $z_2 = y_2^2$ are solutions of a linear differential equation and satisfy

$$a(x)\dot{z}_i = 2b_0(x) + 2b_2(x)z_i, \quad i = 0, 1, 2.$$

Therefore we have

$$\frac{\dot{z}_0(x) - \dot{z}_1(x)}{z_0(x) - z_1(x)} = \frac{\dot{z}_2(x) - \dot{z}_1(x)}{z_2(x) - z_1(x)}.$$

Integrating this equality we obtain

$$z_0(x) = z_1(x) + c(z_2(x) - z_1(x)),$$

with c an arbitrary constant. So the general solution of equation (2) is

$$\begin{aligned} y_0^2(x) &= z_0(x) = z_1(x) + c(z_2(x) - z_1(x)) \\ &= (1-c)y_1^2(x) + cy_2^2(x) = g^2(x)((1-c)\tilde{y}_1^2 + c\tilde{y}_2^2), \end{aligned}$$

with c an arbitrary constant. □

In view of Lemma 3, if $y_1(x), y_2(x)$ are polynomial solutions of equation (4) such that $y_1(x) \not\equiv 0$, $y_2(x) \not\equiv 0$ and $y_2(x) \neq -y_1(x)$ then any other polynomial solution is of the form as in (7) for some appropriate constant c. In particular, $c\tilde{y}_1^2(x)+(1-c)\tilde{y}_2^2(x)$ is a square of a polynomial P. Proceeding exactly as in the proof of Theorem 1 we conclude that equation (4) has at most seven different polynomial solutions 0, $\pm y_1$, $\pm y_2$ and y_0 as in (7).

Example 2. We consider the equivariant polynomial Abel differential equation of second kind (4) with

$$\begin{aligned} a(x) = &\ 2x^4 - 3x^2 + \frac{1}{8}, \\ b_0(x) = &\ \frac{x}{2} - 8x^5, \\ b_2(x) = &\ 4x^3 - 3x. \end{aligned}$$

This equation has the following 7 polynomial solutions

$$\begin{aligned} y_1(x) = &\ 0, \\ y_{2,3}(x) = &\ \pm\left(2x^2 - \frac{1}{2}\right), \\ y_{4,5}(x) = &\ \pm\left(\sqrt{2}x^2 + \frac{1}{2\sqrt{2}}\right), \\ y_{6,7}(x) = &\ \pm 2x. \end{aligned}$$

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