PROPER RATIONAL AND ANALYTIC FIRST INTEGRALS FOR ASYMMETRIC 3-DIMENSIONAL LOTKA-VOLTERRA SYSTEMS

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Abstract. We go beyond in the study of the integrability of the classical model of competition between three species studied by May and Leonard [19], by considering a more realistic asymmetric model. Our results show that there are no global analytic first integrals and we provide all proper rational first integrals of this extended model by classifying its invariant algebraic surfaces.

1. Introduction and statement of the main results

Nonlinear differential equations govern many branches of applied mathematics, physics and sciences in general. A 3–dimensional system with two first integrals whose gradients are linearly independent in $\mathbb{R}^3$ (except perhaps in a zero Lebesgue measure set) is completely solvable in the sense that the intersections of the invariant levels of these two first integrals determine the trajectories of the system. On the other hand, the knowledge of only one first integral does not determine completely the phase portrait of the system but reduces the study of its dynamics by one dimension (i.e. from dimension 3 to dimension 2). Therefore knowing whether there exists a first integral is important in the qualitative theory of differential equations. Different methods exist for studying the existence of first integrals of non–linear ordinary differential equations relying on: the well-known Darboux theory of integrability [8, 18], the so-called Noether symmetries [5], the so-called Lie symmetries [1, 24], the well-known Painlevé analysis [3], the use of Lax pairs [12], the so-called direct method [9, 10], the well-known Carleman embedding procedure [6, 2], the so-called linear compatibility analysis method [25], etc.

In the present paper using the well-known Darboux theory of integrability we determine the existence of first integrals in some given classes for the following asymmetric model that studies the competition among three species and that is the asymmetrization of the initial model used by May and Leonard [19]. This asymmetric model is

$$
\begin{aligned}
\dot{X} &= X(1 - X - a_1 Y - b_1 Z),\\
\dot{Y} &= Y(1 - b_2 X - Y - a_2 Z),\\
\dot{Z} &= Z(1 - a_3 X - b_3 Y - Z),
\end{aligned}
\tag{1}
$$

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