GLOBAL DYNAMICS OF THE INTEGRABLE ARMBRUSTER-GUCKENHEIMER-KIM GALACTIC POTENTIAL

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ABSTRACT. We study the global dynamics of the completely integrable Armbruster-Guckenheimer-Kim galactic potential. In these cases this system has two first integrals H_1 and H_2 independent and in involution. Let I_{h_1} and I_{h_2} be the set of points of the phase space on which H_1 and H_2 take the values h_1 and h_2, respectively. The sets $I_{h_1h_2} = I_{h_1} \cap I_{h_2}$ are invariant by the dynamics. We characterize the global flow on these sets and we describe the foliation of the phase space by the invariant sets $I_{h_1h_2}$.

1. INTRODUCTION

The Armsbruster-Guckenheimer-Kim potential is a galactic potential introduced in [2] that studies the dynamics for the interchanging of nearly nondegenerate modes with square symmetry. They derived the model starting with a normal form given by a system of differential equations which represented the codimension two bifurcation problem. More precisely, the Hamiltonian function that they provided is

$$H(x, p_x, y, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) - \frac{a}{4}(x^2 + y^2)^2 - \frac{b}{2}x^2y^2,$$

where a, b are arbitrary constants. If we add the term $-\omega(xp_y - yp_x)$ then the system describes the dynamics of rotation of a nearly axisymetric galaxy rotating with a constant velocity ω around a fixed axis. The existence of such ω denotes that the rotation of the galaxy must be taken into account when we study the stellar orbits (see [8]). Many studies concerning the integrability and non-integrability of such systems have been done (see for instance [1, 4, 5]) using different techniques such as the Painlevé analysis and the Morales-Ramis theory as well as the study of the existence of periodic orbits which was done in [7]. In particular, it was proved in [5] that if $b = 2a$ or $b = -a$ the

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system is completely integrable but the authors do not describe completely the dynamics of the integrable systems form the point of view of the Liouville-Arnold theorem (see section 2). This is the main aim of this paper.

When $b = 2a$ the Hamiltonian has the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) - \frac{a}{4}(x^2 + y^2)^2 - ax^2y^2.$$

Introducing the new variables

$$u = \frac{1}{\sqrt{2}}(x - y), \; v = \frac{1}{\sqrt{2}}(x + y), \; p_u = \frac{1}{\sqrt{2}}(p_x - p_y), \; p_v = \frac{1}{\sqrt{2}}(p_x + p_y),$$

it can be written as

$$\begin{aligned} H(x, p_x, y, p_y) =& \frac{1}{2}(p_x^2 + p_y^2) + \frac{a}{4}(x^4 + y^4) - \frac{1}{2}(x^2 + y^2) \\ =& \tilde{H}_1(x, p_x) + \tilde{H}_2(y, p_y), \end{aligned}$$

where $a \in \mathbb{R}$, we have renamed the variables (u, v) again as (x, y) and

$$\tilde{H}_1(x, p_x) = \frac{1}{2}p_x^2 + \frac{a}{4}x^4 - \frac{1}{2}x^2, \quad \tilde{H}_2(y, p_y) = \frac{1}{2}p_y^2 + \frac{a}{4}y^4 - \frac{1}{2}y^2.$$

Note that $\tilde{H}_i \colon \mathbb{R}^2 \to \mathbb{R}$ while $H \colon \mathbb{R}^4 \to \mathbb{R}$. In all the paper we will denote by H the Hamiltonian associated to a system with two degrees of freedom and so $H = H(x, p_x, y, p_y) \colon \mathbb{R}^4 \to \mathbb{R}$, $H_i = H_i(x, p_x, y, p_y) \colon \mathbb{R}^4 \to \mathbb{R}$ for $i = 1, \ldots, 4$, and we will denote by $\tilde{H}$ the Hamiltonian associated to a system with one degree of freedom and so $\tilde{H}_1 = \tilde{H}_1(x, p_x) \colon \mathbb{R}^2 \to \mathbb{R}$ and $\tilde{H}_2 = \tilde{H}_2(y, p_y) \colon \mathbb{R}^2 \to \mathbb{R}$.

We observe thta H_1 and H_2 are two first integrals, independent and in involution. Hence, the Hamiltonian system associated to the Hamiltonian H is

$$\dot{x} = p_x, \quad \dot{y} = p_y, \quad \dot{p}_x = -ax^3 + x, \quad \dot{p}_y = -ay^3 + y \tag{1}$$

and it is completely integrable. We recall that H_1 and H_2 are *independent* if the matrix

$$\begin{pmatrix} H_{1x} & H_{1p_x} & H_{1y} & H_{1p_x} \\ H_{2x} & H_{2p_x} & H_{2y} & H_{2p_x} \end{pmatrix}$$

has rank 2 in any point of $\mathbb{R}^4$ except, perhaps in a zero Lebesgue-measure set. As usual $H_{iy} = \partial H_i / \partial y$. Moreover, we say that H_1 and H_2 are in *involution* if their Poisson bracket is zero. Finally, a Hamiltonian system with two degrees of freedom is *completely integrable* if it has two independent first integrals in involution.

Note that the phase space of system (1) is $\mathbb{R}^4$. Since H_1 and H_2 are fist integrals the sets

$$\begin{aligned} I_{h_1} =&\{(x,p_x,y,p_y)\in\mathbb{R}^4 : H_1=h_1\} = \{(x,p_x)\in\mathbb{R}^2 : \tilde H_1=\tilde h_1\}\times\mathbb{R}^2\\ =&I_{\tilde h_1}\times\mathbb{R}^2,\\ I_{h_2} =&\{(x,p_x,y,p_y)\in\mathbb{R}^4 : H_2=h_2\} = \{(y,p_y)\in\mathbb{R}^2 : \tilde H_2=\tilde h_2\}\times\mathbb{R}^2\\ =&\mathbb{R}^2\times I_{\tilde h_2}, \end{aligned}$$

as well as

$$\begin{aligned} I_{h_1h_2} &= \{(x,p_x,y,p_y)\in\mathbb{R}^4 : H_1=h_1, H_2=h_2\}\\ &= \{(x,p_x,y,p_y)\in\mathbb{R}^4 : H_1=h_1\}\cap\{(x,p_x,y,p_y)\in\mathbb{R}^4 : H_2=h_2\}\\ &= I_{h_1}\cap I_{h_2} = \left(I_{\tilde h_1}\times\mathbb{R}^2\right)\cap\left(\mathbb{R}^2\times I_{\tilde h_2}\right)\\ &= I_{\tilde h_1}\times I_{\tilde h_2} \end{aligned}$$

are invariant by the flow of the Hamiltonian system (1). The first objective of this paper is to describe the foliations of the phase space $\mathbb{R}^4$ by the invariant sets I_{h_i} for $i=1,2$ as well as by $I_{h_1h_2}$. The foliations provide a good description of the phase portraits of the Hamiltonian flow (1) when a varies.

When $b=-a$ the Hamiltonian has the form

$$\begin{aligned} H(x,p_x,y,p_y) =&\frac{1}{2}(p_x^2+p_y^2)-\frac{a}{4}(x^4+y^4)+\frac{1}{2}(x^2+y^2)\\ =&\tilde H_3(x,p_x)+\tilde H_4(y,p_y), \end{aligned}$$

where $a\in\mathbb{R}$ with

$$\tilde H_3(x,p_x)=\frac{1}{2}p_x^2-\frac{a}{4}x^4+\frac{1}{2}x^2,\quad \tilde H_4(y,p_y)=\frac{1}{2}p_y^2-\frac{a}{4}y^4+\frac{1}{2}y^2.$$

Note that H_3 and H_4 are two first integrals, independent and in involution. Hence the Hamiltonian system

$$\dot x=p_x,\quad \dot y=p_y,\quad \dot p_x=ax^3-x,\quad \dot p_y=ay^3-y \tag{2}$$

is completely integrable. The sets

$$\begin{aligned} I_{h_3} =&\{(x,p_x,y,p_y)\in\mathbb{R}^4 : H_3=h_3\} = I_{\tilde h_3}\times\mathbb{R}^2,\\ I_{h_4} =&\{(x,p_x,y,p_y)\in\mathbb{R}^4 : H_4=h_4\} = \mathbb{R}^2\times I_{\tilde h_3}, \end{aligned}$$

as well as

$$I_{h_3h_4}=\{(x,p_x,y,p_y)\in\mathbb{R}^4 : H_3=h_3, H_4=h_4\}=I_{h_3}\cap I_{h_4}=I_{\tilde h_3}\times I_{\tilde h_4}$$

are invariant by the flow of the Hamiltonian system (2). The second main objective of the paper is to describe the foliations of $\mathbb{R}^4$ by the invariant sets I_{h_i} for $i=3,4$ and by the invariant sets $I_{h_3h_4}$. Again, these foliations provide a good description of the phase portraits of the Hamiltonian flow (2) when a varies.

The paper is organized as follows. In section 2 we recall the Liouville-Arnold theory for Hamiltonians systems with two degrees of freedom. In section 3 we describe the topology of the sets I_{h_1} (since the study for I_{h_2} is analogous). For doing that and taking into account that $I_{h_1} = I_{\tilde{h}_1} \times \mathbb{R}^2$ we will only describe the topology of the sets $I_{\tilde{h}_1}$ by computing the sets of singular points and critical values for $\tilde{H}_1$ and the Hill regions according to the different values of a and $\tilde{h}_1$. In section 4 we study the topology of the sets $I_{h_1h_2}$. In section 5 we describe the topology of the sets I_{h_3} (again because the study for I_{h_4} is analogous) and recalling that $I_{h_3} = I_{\tilde{h}_3} \times \mathbb{R}^2$ we will only describe the topology of the sets $I_{\tilde{h}_3}$ by computing the sets of singular points and critical values for $\tilde{H}_3$ and the Hill regions according to the different values of a and $\tilde{h}_3$. In section 6 we study the topology of the sets $I_{h_3h_4}$.

2. Integrable Hamiltonian systems

In this section we recall the Liouville-Arnold theorem for the integrable Hamiltonian systems with two degrees of freedom. We recall that a flow defined on the phase space $\mathbb{R}^4$ is complete if its solutions are defined for all time t in $\mathbb{R}$.

Theorem 1. *The Hamiltonian system* (1) *(resp. system* (2)*) defined on the phase space* $\mathbb{R}^4$ *has the Hamiltonians* H_1 *and* H_2 *(resp.* H_3 *and* H_4*) as two independent first integrals in involution. If* $I_{h_1h_2} \neq \emptyset$ *(resp.* $I_{h_3h_4} \neq \emptyset$*) and* (h_1, h_2) *(resp.* (h_3, h_4)*) is a regular value of the map* (H_1, H_2) *(resp.* (H_3, H_4)*) then the following statements hold.*

- (a) $I_{h_1h_2}$ *(resp.* $I_{h_3h_4}$*) is a two-dimensional submanifold of* $\mathbb{R}^4$ *invariant under the flow of system* (1) *(resp. system* (2)*).*
- (b) *If the flow on a connected component* $I^*_{h_1h_2}$ *(resp.* $I^*_{h_3h_4}$*) of* $I_{h_1h_2}$ *(resp.* $I_{h_3h_4}$*) is complete, then* $I^*_{h_1h_2}$ *(resp.* $I^*_{h_3h_4}$*) is diffeomorphic either to the torus* $\mathbb{S}^1 \times \mathbb{S}^1$*, to the cylinder* $\mathbb{S}^1 \times \mathbb{R}$*, or to the plane* $\mathbb{R}^2$
- (c) *Under the assumption of statement (b), the flow on* $I^*_{h_1h_2}$ *(resp. on* $I^*_{h_3h_4}$*) is conjugated to a linear flow either on* $\mathbb{S}^1 \times \mathbb{S}^1$*, or on* $\mathbb{S}^1 \times \mathbb{R}$*, or on* $\mathbb{R}^2$*.*

Note that Theorem 1 does not provide information on the topology of the invariant sets $I_{h_1h_2}$ (resp. $I_{h_3h_4}$) when (h_1h_2) (resp. (h_3h_4)) is not a regular value of the map (H_1, H_2) (resp. (H_3, H_4)), or how the energy levels I_{h_1} or I_{h_2} (resp. I_{h_3} or I_{h_4}) foliate $\mathbb{R}^4$.

In this paper we solve these problems for systems (1) and (2).

3. The topology of the invariant sets I_{h_1}

As explained in the introduction, taking into account that $I_{h_1} = I_{\tilde{h}_1} \times \mathbb{R}^2$ we will restrict all the study to $I_{\tilde{h}_1}$.

A point $(x, p_x) \in \mathbb{R}^2$ is a *singular point* for the map $\tilde{H}_1$ if it is a solution of

$$\frac{\partial \tilde{H}_1}{\partial p_x} = 0, \quad \frac{\partial \tilde{H}_1}{\partial x} = 0.$$

The value $\tilde{h}_1 \in \mathbb{R}$ is a *critical value* for the map $\tilde{H}_1$ if there is some singular point belonging to $\tilde{H}_1^{-1}(\tilde{h}_1) = I_{\tilde{h}_1}$. If $\tilde{h}_1$ is not critical value it is said a *regular value*. It is well-known that if $\tilde{h}_1$ is a regular value of the map $\tilde{H}_1$ then $I_{\tilde{h}_1}$ is a one-dimensional manifold (see [6]).

Note that the singular points for the map $\tilde{H}_1$ are

$$p_x = 0, \quad x(ax^2 - 1) = 0,$$

and so the set of singular points of $\tilde{H}_1$ is $(0, 0)$ if $a \le 0$, and $(0, 0) \cup (0, -1/\sqrt{a}) \cup (0, 1/\sqrt{a})$ if $a > 0$.

We define the Hill region as

$$R_{\tilde{h}_1} = \left\{ x \in \mathbb{R} : \frac{a}{4} x^4 - \frac{x^2}{2} \le \tilde{h}_1 \right\}$$

This is the region of the configuration space $\{x \in \mathbb{R}\}$ where the motion of all orbits of the Hamiltonian system associated to $\tilde{H}_1$ having energy $\tilde{h}_1$ takes place. By $R_{\tilde{h}_1} \approx S$, we denote that $R_{\tilde{h}_1}$ is diffeomorphic to S. We will also denote by

$$P_- = \sqrt{\frac{1 - \sqrt{1 + 4a\tilde{h}_1}}{a}}, \quad P_+ = \sqrt{\frac{1 + \sqrt{1 + 4a\tilde{h}_1}}{a}}$$

have:

(i) $R_{\tilde{h}_1} \approx \mathbb{R}$ if $a = 0$ and $\tilde{h}_1 > 0$,
(ii) $R_{\tilde{h}_1} \approx \mathbb{R}$ but here $\{0\}$, which is a singular point for $\tilde{H}_1$, is in the boundary of the Hill region, if $a = 0$ and $\tilde{h}_1 = 0$,
(iii) $R_{\tilde{h}_1} \approx (-\infty, -\sqrt{-2\tilde{h}_1}] \cup [\sqrt{-2\tilde{h}_1}, \infty)$ if $a = 0$ and $\tilde{h}_1 < 0$,
(iv) $R_{\tilde{h}_1} \approx \mathbb{R}$ if $a < 0$ and $\tilde{h}_1 > 0$,

(v) $R_{\tilde{h}_1} \approx \mathbb{R}$ but here $\{0\}$, which is a singular point for $\tilde{H}_1$, is in the boundary of the Hill region, if $a < 0$ and $\tilde{h}_1 = 0$,
(vi) $R_{\tilde{h}_1} \approx (-\infty, -P_-] \cup [P_-, \infty)$ if $a < 0$ and $\tilde{h}_1 < 0$,
(vii) $R_{\tilde{h}_1} \approx \emptyset$ if $a > 0$ and $\tilde{h}_1 < -1/(4a)$,
(viii) $R_{\tilde{h}_1} \approx \{-\sqrt{\frac{1}{a}}\} \cup \{\sqrt{\frac{1}{a}}\}$ which are two of the singular points for the map $\tilde{H}_1$, if $a > 0$ and $\tilde{h}_1 = -1/(4a)$,
(ix) $R_{\tilde{h}_1} \approx [-P_+, -P_-] \cup [P_-, P_+]$, if $a > 0$ and $\tilde{h}_1 \in (-1/(4a), 0)$,
(x) $R_{\tilde{h}_1} \approx [-\sqrt{\frac{2}{a}}, \sqrt{\frac{2}{a}}]$ but here $\{0\}$, which is a singular point for $\tilde{H}_1$, is in the boundary of the Hill region, if $a > 0$ and $\tilde{h}_1 = 0$,
(xi) $R_{\tilde{h}_1} \approx [-P_+, P_+]$ if $a > 0$ and $\tilde{h}_1 > 0$.

Now we compute the energy levels $I_{\tilde{h}_1}$. From the definition of $I_{\tilde{h}_1}$ we have

$$I_{\tilde{h}_1} = \bigcup_{x \in R_{\tilde{h}_1}} E_x \tag{3}$$

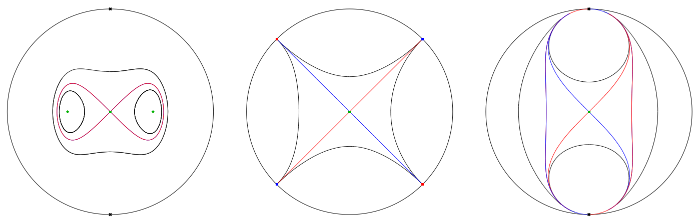
where

$$E_x = \left\{ (x, p_x) \in \mathbb{R}^2 : \frac{p_x^2}{2} + \frac{a}{4}x^4 - \frac{1}{2}x^2 = \tilde{h}_1 \right\}.$$

Clearly for each $x \in \mathbb{R}$ the set E_x is either two points, or one point or the emptyset, if the point x is in the interior of the Hill region $R_{\tilde{h}_1}$, in its boundary, or it does not belong to $R_{\tilde{h}_1}$, respectively. Therefore, from (3) and using the Hill region, the topology of $I_{\tilde{h}_1}$ is:

(i) $I_{\tilde{h}_1} \approx \mathbb{R} \cup \mathbb{R}$ if $a < 0$ and $\tilde{h}_1 \neq 0$,
(ii) $I_{\tilde{h}_1} \approx X$ if $a \leq 0$ and $\tilde{h}_1 = 0$. Here X denotes two straight lines intersecting the origin of the two straight lines,
(iii) $I_{\tilde{h}_1} \approx \emptyset$ if $a > 0$ and $\tilde{h}_1 < -1/(4a)$,
(iv) $I_{\tilde{h}_1} \approx (\pm\sqrt{\frac{1}{a}}, 0)$ which are the two equilibrium points of $\tilde{H}_1$ if $a > 0$ and $\tilde{h}_1 = -1/(4a)$,
(v) $I_{\tilde{h}_1} \approx \mathbb{S}^1 \cup \mathbb{S}^1$ if $a > 0$ and $\tilde{h}_1 \in (-1/(4a), 0)$,
(vi) $I_{\tilde{h}_1} \approx \infty$ if $a > 0$ and $\tilde{h}_1 = 0$. Here ∞ denotes two homoclinic orbits at the origin.
(vii) $I_{\tilde{h}_1} \approx \mathbb{S}^1$ if $a > 0$ and $\tilde{h}_1 < 0$.

See in Figure 1 the phase portraits associated to the Hamiltonian system with Hamiltonian $\tilde{H}_1$ depending on whether $a > 0$, $a = 0$, and $a < 0$. The phase portraits in Figure 1 are drawn in the Poincaré disc, which essentially is a unit closed disc centered at the origin of coordinates with its interior identified to $\mathbb{R}^2$ and with its boundary (the circle $\mathbb{S}^1$) identified with the infinity of $\mathbb{R}^2$, for more details on the Poincaré disc see Chapter 5 of [3].



(a) $a > 0$ (b) $a = 0$ (c) $a < 0$

FIGURE 1. Phase portraits associated to the Hamiltonian system with Hamiltonian $\tilde{H}_1$ depending on whether $a > 0$, $a = 0$ and $a < 0$.

4. The topology of the invariant sets $I_{h_1h_2}$

To obtain $I_{h_1h_2}$ we recall that I_{h_2} is exactly the same as I_{h_1} and that $I_{h_1h_2} = I_{h_1} \cap I_{h_2} = I_{\tilde{h}_1} \times I_{\tilde{h}_2}$. Hence, in Table 1 we have given the description of the invariant sets $I_{h_1h_2}$ for the different values of h_1, h_2 and a

5. The topology of the invariant sets I_{h_3}

As we did for the case H_1, we recall that $I_{h_3} = I_{\tilde{h}_3} \times \mathbb{R}^2$ and so we will study only $I_{\tilde{h}_3}$. The singular points for the map $\tilde{H}_3$ satisfy

$$p_x = 0, \quad x(1 - ax^2) = 0$$

and so they are $(0,0)$ if $a \le 0$ and $(0,0) \cup (0, -1/\sqrt{a}) \cup (0, 1/\sqrt{a})$ if $a > 0$. The Hill region is

$$R_{\tilde{h}_3} = \left\{ y \in \mathbb{R} : -\frac{a}{4}y^4 + \frac{y^2}{2} \le \tilde{h}_3 \right\}$$

and so taking the notation

$$Q_- = \sqrt{\frac{1 - \sqrt{1 - 4a\tilde{h}_3}}{a}}, \quad Q_+ = \sqrt{\frac{1 + \sqrt{1 - 4a\tilde{h}_3}}{a}}$$

we have

(i) $R_{\tilde{h}_3} \approx \emptyset$ if $a = 0$ and $\tilde{h}_3 < 0$,

a	h_1	h_2	$I_{h_1h_2}$
≤ 0	$\neq 0$	$\neq 0$	$(\mathbb{R}\cup\mathbb{R})\times(\mathbb{R}\cup\mathbb{R})$
≤ 0	$\neq 0$	$=0$	$(\mathbb{R}\cup\mathbb{R})\times X$
≤ 0	$=0$	$\neq 0$	$X\times(\mathbb{R}\cup\mathbb{R})$
≤ 0	$=0$	$=0$	$X\times X$
>0	$<-1/(4a)$	$\in\mathbb{R}$	$\emptyset$
>0	$=-1/(4a)$	$<-1/(4a)$	$\emptyset$
>0	$=-1/(4a)$	$=-1/(4a)$	$(\pm\sqrt{\frac{1}{a}},0)\times(\pm\sqrt{\frac{1}{a}},0)$
>0	$=-1/(4a)$	$\in(-1/(4a),0)$	$(\pm\sqrt{\frac{1}{a}},0)\times(\mathbb{S}^1\cup\mathbb{S}^1)$
>0	$=-1/(4a)$	$=0$	$(\pm\sqrt{\frac{1}{a}},0)\times\infty$
>0	$=-1/(4a)$	<0	$(\pm\sqrt{\frac{1}{a}},0)\times\mathbb{S}^1$
>0	$\in(-1/(4a),0)$	$<-1/(4a)$	$\emptyset$
>0	$\in(-1/(4a),0)$	$=-1/(4a)$	$(\mathbb{S}^1\cup\mathbb{S}^1)\times(\pm\sqrt{\frac{1}{a}},0)$
>0	$\in(-1/(4a),0)$	$\in(-1/(4a),0)$	$(\mathbb{S}^1\cup\mathbb{S}^1)\times(\mathbb{S}^1\cup\mathbb{S}^1)$
>0	$\in(-1/(4a),0)$	$=0$	$(\mathbb{S}^1\cup\mathbb{S}^1)\times\infty$
>0	$\in(-1/(4a),0)$	<0	$(\mathbb{S}^1\cup\mathbb{S}^1)\times\mathbb{S}^1$
>0	$=0$	$<-1/(4a)$	$\emptyset$
>0	$=0$	$=-1/(4a)$	$\infty\times(\pm\sqrt{\frac{1}{a}},0)$
>0	$=0$	$\in(-1/(4a),0)$	$\infty\times(\mathbb{S}^1\cup\mathbb{S}^1)$
>0	$=0$	$=0$	$\infty\times\infty$
>0	$=0$	<0	$\infty\times\mathbb{S}^1$
>0	<0	$<-1/(4a)$	$\emptyset$
>0	<0	$=-1/(4a)$	$\mathbb{S}^1\times(\pm\sqrt{\frac{1}{a}},0)$
>0	<0	$\in(-1/(4a),0)$	$\mathbb{S}^1\times(\mathbb{S}^1\cup\mathbb{S}^1)$
>0	<0	$=0$	$\mathbb{S}^1\times\infty$
>0	<0	<0	$\mathbb{S}^1\times\mathbb{S}^1$

TABLE 1. The invariant sets $I_{h_1h_2}$ for the different values of h_1, h_2 and a

(ii) $R_{\tilde{h}_3}\approx\{0\}$ if $a=0$ and $\tilde{h}_3=0$,then
(iii) $R_{\tilde{h}_3}\approx[-\sqrt{2\tilde{h}_3},\sqrt{2\tilde{h}_3}]$ if $a=0$ and $\tilde{h}_3>0$ then
(iv) $R_{\tilde{h}_3}\approx\emptyset$ if $a<0$ and $\tilde{h}_3<0$,
(v) $R_{\tilde{h}_3}\approx\{0\}$ if $a<0$ and $\tilde{h}_3=0$,
(vi) $R_{\tilde{h}_3}\approx[-Q_+,Q_-]$ if $a<0$ and $\tilde{h}_3>0$,
(vii) $R_{\tilde{h}_3}\approx\mathbb{R}$ if $a>0$ and $h_3>1/(4a)$,

(viii) $R_{\tilde{h}_3} \approx \mathbb{R}$, but here $\{\pm\sqrt{\frac{1}{a}}\}$, which are singular points for $\tilde{H}_1$, are in the boundary of the Hill region, if $a > 0$ and $\tilde{h}_3 = 1/(4a)$,

(ix) $R_{\tilde{h}_3} \approx (-\infty, -Q_+] \cup [-Q_-, Q_-] \cup [Q_+, +\infty)$ if $a > 0$ and $\tilde{h}_3 \in (0, 1/(4a))$,

(x) $R_{\tilde{h}_3} \approx \mathbb{R}$, but here $\{0\}$, which is a singular point for $\tilde{H}_1$, is in the boundary of the Hill region, if $a > 0$ and $\tilde{h}_3 = 0$,

(xi) $R_{\tilde{h}_3} \approx \big(-\infty, -Q_+] \cup [Q_+, +\infty)$ if $a > 0$ and $\tilde{h}_3 < 0$.

Now we compute the energy levels $I_{\tilde{h}_3}$. From the definition of $I_{\tilde{h}_3}$ we have

$$I_{\tilde{h}_3} = \cup_{y \in R_{\tilde{h}_3}} E_y \tag{4}$$

where

$$E_y = \left\{ (y, p_y) \in \mathbb{R}^2 : \frac{p_y^2}{2} - \frac{a}{4} y^4 + \frac{1}{2} y^2 = \tilde{h}_3 \right\}.$$

Clearly for each $y \in \mathbb{R}$ the set E_y is either two points, or one point or the emptyset, if the point y is in the interior of the Hill region $R_{\tilde{h}_3}$, in its boundary, or it does not belong to $R_{\tilde{h}_3}$, respectively. Therefore, from (4) and using the Hill region, the topology of $I_{\tilde{h}_3}$ is:

(i) $I_{\tilde{h}_3} \approx \emptyset$ if $a \leq 0$ and $\tilde{h}_3 < 0$,

(ii) $I_{\tilde{h}_3} \approx \{(0,0)\}$ if $a \leq 0$ and $\tilde{h}_3 = 0$,

(iii) $I_{\tilde{h}_3} \approx \mathbb{S}^1$ $a \leq 0$ and $\tilde{h}_3 > 0$,

(iv) $I_{\tilde{h}_3} \approx \mathbb{R} \cup \mathbb{R}$ if $a > 0$ and $\tilde{h}_3 > 1/(4a)$,

(v) $I_{\tilde{h}_3} \approx P$ if $a > 0$ and $\tilde{h}_3 = 1/(4a)$. Here P denotes two curves with the shape of a parabola intersecting in two different points (the points are the two singular points),

(vi) $I_{\tilde{h}_3} \approx \mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R}$ if $a > 0$ and $\tilde{h}_3 \in (0, 1/(4a))$,

(vii) $I_{\tilde{h}_3} \approx \mathbb{R} \cup \{(0,0)\} \cup \mathbb{R}$ if $a > 0$ and $\tilde{h}_3 = 0$,

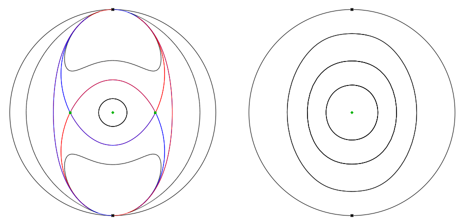
(viii) $I_{\tilde{h}_3} \approx \mathbb{R} \cup \mathbb{R}$ if $a > 0$ and $\tilde{h}_3 < 0$.

See the phase portrait associated to $\tilde{H}_3$ depending on whether $a > 0$, $a = 0$, or $a < 0$.

See in Figure 2 the phase portraits associated to the Hamiltonian system with Hamiltonian $\tilde{H}_3$ depending on whether $a > 0$ and $a \leq 0$.

6. The topology of the invariant sets $I_{h_3 h_4}$

To obtain $I_{h_3 h_4}$ we recall that I_{h_4} is exactly the same as I_{h_3} and that $I_{h_3 h_4} = I_{h_3} \cap I_{h_4} = I_{\tilde{h}_3} \times I_{\tilde{h}_4}$. Hence, in Table 2 we have given the



(a) $a > 0$ (b) $a \leq 0$

Figure 2. Phase portraits associated to the Hamiltonian system with Hamiltonian $\tilde{H}_3$ depending on whether $a > 0$ or $a \leq 0$.

description of the invariant sets $I_{h_3 h_4}$ for the different values of h_3, h_4 and a.

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a	h_1	h_2	$I_{h_1h_2}$
≤ 0	< 0	$\in \mathbb{R}$	$\emptyset$
≤ 0	$= 0$	< 0	$\emptyset$
≤ 0	$= 0$	$= 0$	$\{(0,0)\} \times \{(0,0)\}$
≤ 0	$= 0$	> 0	$\{(0,0)\} \times \mathbb{S}^1$
≤ 0	> 0	< 0	$\emptyset$
≤ 0	> 0	$= 0$	$\mathbb{S}^1 \times \{(0,0)\}$
≤ 0	> 0	> 0	$\mathbb{S}^1 \times \mathbb{S}^1$
≤ 0	$> 1/(4a)$	$> 1/(4a)$	$(\mathbb{R} \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{R})$
≤ 0	$> 1/(4a)$	$= 1/(4a)$	$(\mathbb{R} \cup \mathbb{R}) \times P$
≤ 0	$> 1/(4a)$	$\in (0, 1/(4a))$	$(\mathbb{R} \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R})$
≤ 0	$> 1/(4a)$	$= 0$	$(\mathbb{R} \cup \mathbb{R}) \times (\mathbb{R} \cup \{(0,0)\} \cup \mathbb{R})$
≤ 0	$> 1/(4a)$	< 0	$(\mathbb{R} \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{R})$
≤ 0	$= 1/(4a)$	$> 1/(4a)$	$P \times (\mathbb{R} \cup \mathbb{R})$
≤ 0	$= 1/(4a)$	$= 1/(4a)$	$P \times P$
≤ 0	$= 1/(4a)$	$\in (0, 1/(4a))$	$P \times (\mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R})$
≤ 0	$= 1/(4a)$	$= 0$	$P \times (\mathbb{R} \cup \{(0,0)\} \cup \mathbb{R})$
≤ 0	$= 1/(4a)$	< 0	$P \times (\mathbb{R} \cup \mathbb{R})$
≤ 0	$\in (0, 1/(4a))$	$> 1/(4a)$	$(\mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{R})$
≤ 0	$\in (0, 1/(4a))$	$= 1/(4a)$	$(\mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R}) \times P$
≤ 0	$\in (0, 1/(4a))$	$\in (0, 1/(4a))$	$(\mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R})$
≤ 0	$\in (0, 1/(4a))$	$= 0$	$(\mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R}) \times (\mathbb{R} \cup \{(0,0)\} \cup \mathbb{R})$
≤ 0	$\in (0, 1/(4a))$	< 0	$(\mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{R})$
≤ 0	$= 0$	$> 1/(4a)$	$(\mathbb{R} \cup \{(0,0)\} \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{R})$
≤ 0	$= 0$	$= 1/(4a)$	$(\mathbb{R} \cup \{(0,0)\} \cup \mathbb{R}) \times P$
≤ 0	$= 0$	$\in (0, 1/(4a))$	$(\mathbb{R} \cup \{(0,0)\} \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R})$
≤ 0	$= 0$	$= 0$	$(\mathbb{R} \cup \{(0,0)\} \cup \mathbb{R}) \times (\mathbb{R} \cup \{(0,0)\} \cup \mathbb{R})$
≤ 0	$= 0$	< 0	$(\mathbb{R} \cup \{(0,0)\} \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{R})$
≤ 0	< 0	$> 1/(4a)$	$(\mathbb{R} \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{R})$
≤ 0	< 0	$= 1/(4a)$	$(\mathbb{R} \cup \mathbb{R}) \times P$
≤ 0	< 0	$\in (0, 1/(4a))$	$(\mathbb{R} \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{S}^1 \cup \mathbb{R})$
≤ 0	< 0	$= 0$	$(\mathbb{R} \cup \mathbb{R}) \times (\mathbb{R} \cup \{(0,0)\} \cup \mathbb{R})$
≤ 0	< 0	< 0	$(\mathbb{R} \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{R})$

TABLE 2. The invariant sets $I_{h_3h_4}$ for the different values of h_3, h_4 and a

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